/ /

(II) Entre dois dipolos

- $U = \frac{1}{2}\sum_i q_i V(\vec{r}_i)$

- $V_{dip} \simeq \frac{\vec{p}\cdot\vec{r}}{r^3}$ $\Rightarrow U \simeq \frac{Q}{2}\left(\underbrace{\frac{\vec{p}_1\cdot\vec{r}}{r^3}}_{V(\vec{r})} - \underbrace{\frac{\vec{p}_1\cdot(\vec{r}-\vec{d})}{r^3}}_{V(\vec{r}-\vec{d})}\right) \Rightarrow \boxed{U \simeq \frac{\vec{p}_1\cdot\vec{p}_2}{r^3}}$

- $\vec{p} = Q\vec{d}$

OBS: interpretar o elétron como partícula singular falha para a região $r < a$. Da junção Relat. Restrita + Mec. Quântica surge:

$$\begin{cases} a = l_c = \frac{\hbar}{m_e c} \\ \dot{E} \neq 0 \end{cases}, \text{como } \frac{e^2}{2a} < m_e c^2 \Rightarrow \frac{\alpha}{2} < 1, \quad \alpha = \frac{e^2}{\hbar c} = \frac{1}{137}$$

estrutura fina

Condutores

↳ materiais que permitem a movimentação das cargas em excesso distribuindo-as em sua superfície externa, portanto:

- Dentro do condutor: $\vec{E}_{int} = 0$ $\begin{cases} \vec{E} = -\vec{\nabla}\phi \Rightarrow \phi_{int} = \text{constante} \\ \vec{\nabla}\cdot\vec{E} = 4\pi\rho \Rightarrow \rho_{int} = 0 \end{cases}$

- Na superfície: $\vec{E} \perp d\vec{s} \Rightarrow \vec{E}_\perp = 4\pi\sigma\hat{n}$, $d\vec{s} = dS\,\hat{n}$

Nova classe de problemas com $\nabla^2\phi = 0$ e condições de contorno nas bordas

(N) Carga fixa "Neumann"

(D) Potencial fixo "Dirichlet"

Teorema de Unicidade da Solução

↳ ∀ condição de contorno $\nabla^2\phi = 0$ tem solução única

Por absurdo: Se ϕ_1, ϕ_2 são duas soluções com as mesmas condições de contorno, defina $f \equiv \phi_1 - \phi_2$ ∴ $\nabla^2 f = 0$

$$\Rightarrow \sum_i \int_{S_i} f \vec{\nabla} f \, d\vec{s} = \int_V \vec{\nabla}\cdot(f\vec{\nabla}f)\, d^3r = \int_V \left(|\vec{\nabla} f|^2 + f \nabla^2 f \right) d^3r$$

$$\Leftrightarrow \int_V |\vec{\nabla} f|^2 \, d^3r = \sum_i \int_{S_i} f \vec{\nabla} f \, d\vec{s} = 0$$

pois ou (D) $f = 0$

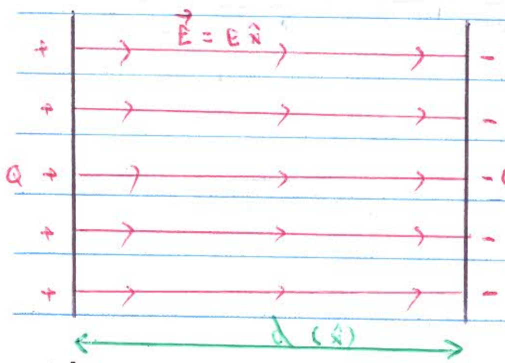
(N) $\vec{\nabla} f = 0$

$$\Rightarrow \vec{\nabla} f = 0 \Rightarrow f = cte \Leftrightarrow \phi_1 = \phi_2 + \text{constante}$$

Irrelevante

Exemplos

① Capacitor de placas Paralelas



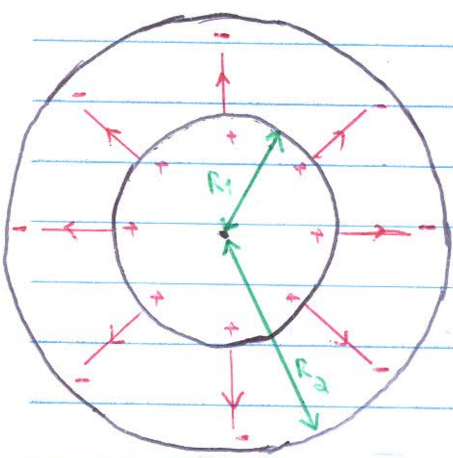
- $\vec{E} = 4\pi\sigma\hat{n}$ • $\sigma = \frac{Q}{A}$ • $C = \frac{Q}{V}$

$$\Rightarrow E = -\frac{d\phi}{dx} \Rightarrow \phi = -Ex + k$$

$$\Rightarrow V = \phi(0) - \phi(d) = Ed \Rightarrow V = 4\pi \frac{Q}{A} \cdot d$$

$$\Leftrightarrow \boxed{C = \frac{A}{4\pi d}} \quad e \quad \boxed{U = \frac{Q^2}{2C}}$$

② Capacitor esférico



- $\vec{E} = -\nabla\phi$ • $\sigma = \frac{Q}{4\pi R^2}$ • $\vec{E} = 4\pi\sigma\hat{n}$

$$\Rightarrow \vec{E} = \frac{Q}{R^2}\hat{n} \Rightarrow \phi = -\frac{Q}{R}\hat{r}$$

$$\Rightarrow V = \phi(R_1) - \phi(R_2) = +Q\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$\Rightarrow \frac{Q}{V} = C = \frac{R_1 R_2}{R_2 - R_1}$$

Teorema de Green

$$\oint_{\partial V} \vec{A}\cdot\hat{n}\, ds = \int_V \vec{\nabla}\cdot\vec{A}\, d^3x \quad , \text{ fazendo } \vec{A} = \varphi\vec{\nabla}\psi \quad \begin{cases} \vec{\nabla}\cdot\vec{A} = \vec{\nabla}\varphi\cdot\vec{\nabla}\psi + \varphi\nabla^2\psi \\ \vec{A}\cdot\hat{n} = \varphi\vec{\nabla}\psi\cdot\hat{n} = \varphi\dfrac{\partial\psi}{\partial n} \end{cases}$$

$$\Rightarrow \int_V \left(\varphi\nabla^2\psi + \vec{\nabla}\varphi\cdot\vec{\nabla}\psi\right) d^3x = \oint_{\partial V} \varphi\frac{\partial\psi}{\partial n}\, ds$$

fazendo $\varphi \leftrightarrow \psi$

$$\ominus \quad \int_V \left(\psi\nabla^2\varphi + \vec{\nabla}\psi\cdot\vec{\nabla}\varphi\right) d^3x = \int_{\partial V} \psi\frac{\partial\varphi}{\partial n}\, ds$$

$$\Rightarrow \int_V \left(\varphi\nabla^2\psi - \psi\nabla^2\varphi\right) d^3x = \oint_{\partial V}\left(\varphi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\varphi}{\partial n}\right) ds \quad , \text{ fazendo } \begin{cases} \varphi = \phi \Rightarrow \nabla^2\varphi = -4\pi\rho \\ \psi = \dfrac{1}{|\vec{r}-\vec{r}'|} \Rightarrow \nabla^2\psi = -4\pi\delta^3(r-r') \end{cases}$$

$$\Rightarrow \int_V \left(\overbrace{\phi\cdot -4\pi\,\delta^3(\vec{r}-\vec{r}')}^{-4\pi\phi(\vec{r})} - \frac{1}{|\vec{r}-\vec{r}'|}\cdot -4\pi\rho\right) d^3x' = \oint_{\partial V}\left(\phi\cdot\frac{\partial}{\partial n'}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) - \frac{1}{|\vec{r}-\vec{r}'|}\cdot\frac{\partial\phi}{\partial n'}\right) ds'$$

$$\Rightarrow \phi(\vec{r}) = \int_V \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}\, d^3x' + \frac{1}{4\pi}\oint_{\partial V}\left(\frac{1}{|\vec{r}-\vec{r}'|}\frac{\partial\phi(\vec{r}')}{\partial n'} - \phi(\vec{r}')\frac{\partial}{\partial n'}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)\right) ds'$$

Nova Função de Green

$$G(\vec{r},\vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|} + F(\vec{r},\vec{r}') \quad \text{tal que} \quad \nabla^2 F(\vec{r},\vec{r}') = 0$$

$$\Rightarrow \phi(\vec{r}) = \int_V \rho(\vec{r}')\, G(\vec{r},\vec{r}')\, d^3x' + \frac{1}{4\pi}\oint_{\partial V}\left(G(\vec{r},\vec{r}')\underset{\downarrow \atop \text{N}}{\frac{\partial\phi(\vec{r}')}{\partial n'}} - \underset{\downarrow \atop \text{D}}{\phi(\vec{r}')}\frac{\partial G(\vec{r},\vec{r}')}{\partial n'}\right) ds'$$

- D em ∂V: escolha F tal que $G_D(\vec{r},\vec{r}') = 0$
- N em ∂V: escolha $\oint_{\partial V} \frac{\partial G(\vec{r},\vec{r}')}{\partial n'}\, ds' = -4\pi \Rightarrow \frac{\partial G(\vec{r},\vec{r}')}{\partial n'} = \frac{-4\pi}{S}$

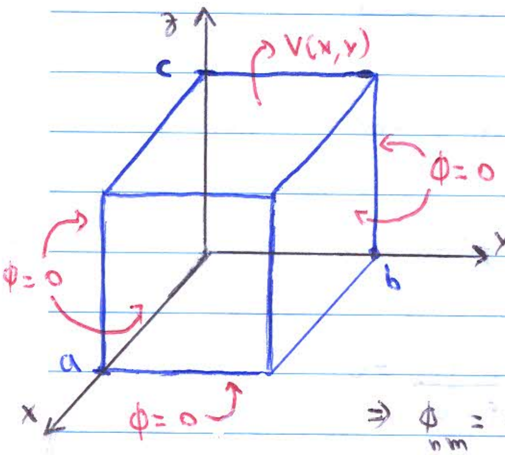
$$\Rightarrow \phi(\vec{r}) = \langle\phi\rangle_{\partial V} + \int_V \rho(\vec{r}')\, G_N(\vec{r},\vec{r}')\, d^3x + \frac{1}{4\pi}\oint_{\partial V}\frac{\partial\phi(\vec{r}')}{\partial n'}\, G_N(\vec{r},\vec{r}')\, ds'$$

Separação de variáveis em coordenadas Cartesianas

$$\begin{cases} \nabla^2\phi = \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} + \dfrac{\partial^2\phi}{\partial z^2} = 0 \\ \phi(x,y,z) = X(x)\cdot Y(y)\cdot Z(z) \end{cases} \Rightarrow \underbrace{\frac{1}{X}\frac{\partial^2 X}{\partial x^2}}_{f(x)} + \underbrace{\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}}_{g(y)} + \underbrace{\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}}_{h(z)} = 0$$

$$\begin{cases} f(x) = -\alpha^2 \\ g(y) = -\beta^2 \\ h(z) = \alpha^2+\beta^2 \end{cases} \Rightarrow \begin{cases} X(x) = e^{\pm i\alpha x} \\ Y(y) = e^{\pm i\beta y} \\ Z(z) = e^{\pm\sqrt{\alpha^2+\beta^2}\, z} \end{cases} \Rightarrow \phi(x,y,z) = e^{\pm i(\alpha x+\beta y) \pm \sqrt{\alpha^2+\beta^2}\, z}$$

- Exemplo: caixa de potencial fixo



$$\text{se} \begin{cases} x=0 \\ y=0 \\ z=0 \end{cases} \Rightarrow \phi = 0 \Leftrightarrow \begin{cases} X(x) = \operatorname{sen}(\alpha x) \\ Y(y) = \operatorname{sen}(\beta y) \\ Z(z) = \operatorname{sh}(\sqrt{\alpha^2+\beta^2}\, z) \end{cases}$$

$$\text{se} \begin{cases} x=a \\ y=b \end{cases} \Rightarrow \phi = 0 \Rightarrow \begin{cases} \alpha a = n\pi \\ \beta b = m\pi \end{cases} \Rightarrow \begin{cases} \alpha_n = \dfrac{n\pi}{a} \\ \beta_m = \dfrac{m\pi}{b} \end{cases}$$

$$\Rightarrow \phi_{nm} = \operatorname{sen}(\alpha_n x)\cdot\operatorname{sen}(\beta_m y)\cdot\operatorname{sh}(\gamma_{nm} z), \quad \gamma_{nm} = \sqrt{\frac{n^2}{a^2}+\frac{m^2}{b^2}}$$

- Princípio da superposição e série de Fourier

$$\begin{cases} \phi(x,y,z) = \sum_{n,m} A_{nm}\,\phi_{nm}(x,y,z) \\ \phi(x,y,c) = V(x,y) \end{cases} \Rightarrow V(x,y) = \sum_{n,m} A_{nm}\operatorname{sen}(\alpha_n x)\operatorname{sen}(\beta_m y)\operatorname{sh}(\gamma_{nm} c)$$

onde, usando a ortogonalidade das funções trigonométricas, calcula-se os A_{nm}

$$\Rightarrow A_{nm} = \frac{4}{ab\,\operatorname{sh}(\gamma_{nm} c)} \int_0^a\int_0^b V(x,y)\operatorname{sen}(\alpha_n x)\operatorname{sen}(\beta_m y)\, dy\, dx$$

Separação de variáveis em coordenadas esféricas

$$\begin{cases} \nabla^2 \phi = \dfrac{1}{r} \dfrac{\partial^2}{\partial r^2}(r\phi) + \dfrac{1}{r^2 \operatorname{sen}\theta} \dfrac{\partial}{\partial \theta}\left(\operatorname{sen}\theta \dfrac{\partial \phi}{\partial \theta}\right) + \dfrac{1}{r^2 \operatorname{sen}^2\theta} \dfrac{\partial^2 \phi}{\partial \varphi^2} = 0 \\ \phi(r,\theta,\varphi) = \dfrac{U(r)}{r} \cdot P(\theta) \cdot Q(\varphi) \end{cases}$$

$$\Rightarrow \frac{r^2 \operatorname{sen}^2\theta}{U(r)} \frac{d^2 U(r)}{dr^2} + \overbrace{\frac{\operatorname{sen}\theta}{P(\theta)} \frac{d}{d\theta}\left(\operatorname{sen}\theta \frac{dP(\theta)}{d\theta}\right)}^{f(\theta)} + \overbrace{\frac{1}{Q(\varphi)} \frac{d^2 Q(\varphi)}{d\varphi^2}}^{g(\varphi)} = 0$$

• $\begin{cases} g(\varphi) = \alpha^2 \\ Q(\varphi) = Q(\varphi + 2\pi) \end{cases} \Rightarrow Q(\varphi) = e^{\pm\alpha\varphi} = e^{\pm\alpha(\varphi+2\pi)} \Leftrightarrow \alpha = \pm i m \Rightarrow Q(\varphi) = e^{im\varphi}$

inteiro positivo

Portanto $\dfrac{d^2 Q(\varphi)}{d\varphi^2} = -m^2 Q(\varphi) \Rightarrow \dfrac{1}{Q(\varphi)} \dfrac{d^2 Q(\varphi)}{d\varphi^2} = -m^2$, daí

$$\Rightarrow \frac{r^2 \operatorname{sen}^2\theta}{U(r)} \frac{d^2 U(r)}{dr^2} + \frac{\operatorname{sen}\theta}{P(\theta)} \frac{d}{d\theta}\left(\operatorname{sen}\theta \frac{dP(\theta)}{d\theta}\right) - m^2 = 0 \qquad \times \frac{P(\theta)}{\operatorname{sen}^2\theta}$$

$$P(\theta) \cdot \underbrace{\frac{r^2}{U(r)} \cdot \frac{d^2 U(r)}{dr^2}}_{\lambda} + \frac{1}{\operatorname{sen}\theta} \frac{d}{d\theta}\left(\operatorname{sen}\theta \frac{dP(\theta)}{d\theta}\right) - \frac{m^2 P(\theta)}{\operatorname{sen}^2\theta} = 0$$

$$\Rightarrow \begin{cases} \dfrac{1}{\operatorname{sen}\theta} \dfrac{d}{d\theta}\left(\operatorname{sen}\theta \dfrac{dP(\theta)}{d\theta}\right) + \left(\lambda - \dfrac{m^2}{\operatorname{sen}^2\theta}\right) P(\theta) = 0 \quad \text{(I)} \\ \dfrac{d^2 U(r)}{dr^2} - \dfrac{\lambda}{r} U(r) = 0 \quad \text{(II)} \end{cases}$$

Em (I):

$$\frac{1}{\operatorname{sen}\theta}\left(\cos\theta \frac{dP(\theta)}{d\theta} + \operatorname{sen}\theta \cdot \frac{d^2 P(\theta)}{d\theta^2}\right) + \left(\lambda - \frac{m^2}{\operatorname{sen}^2\theta}\right) P(\theta) = 0$$

$$\Rightarrow \frac{d^2 P(\theta)}{d\theta^2} + \operatorname{cotg}(\theta) \frac{dP(\theta)}{d\theta} + \left(\lambda - \frac{m^2}{\operatorname{sen}^2\theta}\right) P(\theta) = 0$$

/ /

fazendo $\cos\theta \equiv x$, $x \in [-1,1] \Rightarrow \begin{cases} x_\theta = -\operatorname{sen}\theta \Rightarrow x_{\theta\theta} = -x \\ P_\theta = x_\theta P' = -\operatorname{sen}\theta\, P' \\ P_{\theta\theta} = (x_\theta P')_\theta = (1-x^2) P'' - x P' \end{cases}$, daí

$$\Rightarrow (1-x^2) P'' - x P' + \frac{\cos\theta}{\operatorname{sen}\theta} \cdot -\operatorname{sen}\theta\, P' + \left(\lambda - \frac{m^2}{1-x^2}\right) P = 0$$

$$\Rightarrow (1-x^2) P'' - 2x P' + \left(\lambda - \frac{m^2}{1-x^2}\right) P = 0$$ "Equação de Legendre"

• Se $m=0$:

singularidade regular em $x=1$

$$\begin{cases} P'' + p(x) P' + q(x) P = 0 \\ p(x) = \dfrac{-2x}{1-x^2} \quad \text{e} \quad q(x) = \dfrac{\lambda}{1-x^2} \end{cases} \Rightarrow \begin{cases} p_0 = \lim\limits_{x\to 1} (x-1)\, p(x) = 1 \\ q_0 = \lim\limits_{x\to 1} (x-1)^2 q(x) = 0 \end{cases}$$

aceite

$$\Rightarrow \rho^2 + (p_0 - 1)\rho + q_0 = 0 \longrightarrow \rho^2 = 0 \Leftrightarrow \rho_1 = \rho_2 = 0$$

Portanto a solução geral tem a forma $P = \sum_n c_n (x-1)^n$ "aceite"

fazendo $\xi = (x-1)$ e substituindo na equação de Legendre:

$$\sum_n \left[-\xi(\xi+2) n(n-1) c_n \xi^{n-2} - 2(\xi+1) c_n \xi^{n-1} + \lambda c_n \xi^n \right] = 0$$

$$\Rightarrow \sum_n c_n \left(\xi^n (n^2+n-\lambda) + 2n^2 \xi^{n-1} \right) \Leftrightarrow \sum_n \left[(n^2+n-\lambda) c_n + 2(n+1)^2 c_{n+1} \right] \xi^n = 0$$

Fórmula de Recursão

$$\Rightarrow c_{n+1} = \frac{\lambda - n^2 - n}{2(n+1)^2} c_n, \quad \lambda \equiv \nu(\nu+1) \Rightarrow c_{n+1} = \frac{-(n-\nu)(n+\nu+1)}{2(n+1)^2} c_n$$

• na simbologia $\begin{cases} (a)_0 = 1 \\ (a)_n = a \cdot (a+1) \dots (a+n-1) \end{cases}$ $\Rightarrow c_n = \left(-\frac{1}{2}\right)^n \frac{(-\nu)_n (\nu+1)_n}{(1)_n} \cdot \frac{1}{n!} c_0$

tilibra

• Solução Final $\longrightarrow$ $f(x) = \sum_n \frac{(-\nu)_n (\nu+1)_n}{(1)_n} \cdot \frac{1}{n!} \left(\frac{1-x}{2}\right)^n$

• Série Hipergeométrica $\longrightarrow$ $F(a,b,c,z) = \sum_n \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{1}{n!} z^n$

Interpreta-se $f(x) = F\left(-\nu, \nu+1, 1, \frac{1-x}{2}\right)$

Tomando $\nu = l \in \mathbb{Z}^+$ a solução é finita para $x = \pm 1$

$\Rightarrow P_l(x) \equiv F\left(-l, l+1, 1, \frac{1-x}{2}\right)$