Energia do Campo Eletromagnético

$$U = \frac{1}{8\pi}\int_V \left(\vec{E}^2 + \vec{B}^2\right) d^3x \Rightarrow \frac{dU}{dt} = \frac{1}{4\pi}\int_V \left(\vec{E}\cdot\frac{\partial\vec{E}}{\partial t} + \vec{B}\cdot\frac{\partial\vec{B}}{\partial t}\right) d^3x$$

$$\Rightarrow \frac{dU}{dt} = \frac{1}{4\pi}\int_V \vec{E}\left(\vec{\nabla}\times\vec{B} - 4\pi\vec{j}\right) + \vec{B}\left(-\vec{\nabla}\times\vec{E}\right) d^3x$$

$$= \int_V \frac{1}{4\pi}\left(\vec{E}\cdot\vec{\nabla}\times\vec{B} - \vec{B}\cdot\vec{\nabla}\times\vec{E}\right) - \vec{E}\cdot\vec{j}\; d^3x$$

$$= \frac{1}{4\pi}\int_V -\vec{\nabla}\cdot\left(\vec{E}\times\vec{B}\right) d^3x - \int_V \vec{E}\cdot\vec{j}\, d^3x$$

$$\boxed{\frac{dU}{dt} + \int_V \vec{E}\cdot\vec{j}\, d^3x = -\frac{1}{4\pi}\int_S \left(\vec{E}\times\vec{B}\right)\cdot d\vec{S}}$$

variação da energia do campo

Trabalho feito sobre a carga com velocidade $\vec{v}$

Vetor de Poynting $\vec{S} \equiv \vec{E}\times\vec{B}$ associado à transmissão de energia

Teorema de Poynting em forma local

densidade de energia

$$\frac{\partial \varepsilon}{\partial t} + \vec{\nabla}\cdot\vec{S} = 0 \quad \text{onde} \quad \vec{S} \equiv \vec{E}\times\vec{B}$$

vetor de Poynting

Tensor Energia - Momento

$$\begin{cases} H = \int \mathcal{H}\, dV \\ L = \int \mathcal{L}\, dV \end{cases} \Rightarrow H = p\dot{q} - L \longrightarrow \mathcal{H} = \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial t}\right)}\cdot\frac{\partial\phi}{\partial t} - \mathcal{L}$$

$$\Rightarrow \boxed{T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\cdot\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L}}$$

"seguindo ideias de promoção e completamento"

Construção baseada no teorema de Noether

$$S = \int\int_V \mathcal{L}(\phi, \partial_\mu \phi)\, d^3x\, dt \longrightarrow \begin{cases} x \to x' \\ \phi(x) \to \phi'(x') \end{cases} \Leftrightarrow \begin{cases} x' = x'(x) \\ \phi'(x') = f(\phi(x)) \end{cases}$$

$$\Rightarrow S' = \int \mathcal{L}(\phi'(x), \partial_\mu(\phi'(x)))\, d^4x \longrightarrow \int \mathcal{L}(\phi'(x'), \partial'_\mu \phi'(x'))\, d^4x'$$

$$\Rightarrow S' = \int \mathcal{L}(f(\phi(x)), \partial'_\mu f(\phi(x)))\, d^4x' = \int \underbrace{\left|\frac{\partial x'}{\partial x}\right|}_{\text{Jacobiano}} \cdot \mathcal{L}\Big(f(\phi(x)), \underbrace{\frac{\partial x^\nu}{\partial x'^\mu} \cdot \partial_\nu f(\phi(x))}_{\text{cadeia}}\Big)\, d^4x$$

Exemplo 1: translação

$$\begin{cases} x'^\mu = x^\mu + a^\mu \\ \phi'(x+a) = \phi(x) \end{cases} \Rightarrow \begin{cases} \dfrac{\partial x^\nu}{\partial x'^\mu} = \delta^\nu_\mu \\ f(\phi(x)) = 1 \end{cases} \Rightarrow S = S'$$

Exemplo 2: Lorentz

$$\begin{cases} x'^\mu = \Lambda^\mu{}_\nu x^\nu \\ \phi'(\Lambda x) = L_\Lambda \phi(x) \end{cases} \Rightarrow \begin{cases} \dfrac{\partial x^\nu}{\partial x'^\mu} = \Lambda^{-1} \\ f(\phi(x)) = L_\Lambda \phi(x) \end{cases} \Rightarrow S' = \int \overbrace{\left|\frac{\partial x'}{\partial x}\right|}^{\text{Lorentz tem norma 1}} \mathcal{L}(L_\Lambda \phi, \Lambda^{-1}\partial(L_\Lambda \phi))\, d^4x$$

- Se $\phi = \varphi$ (escalar) $\Rightarrow \begin{cases} S = S' = \int \mathcal{L}(\varphi, \Lambda^{-1}\partial\varphi)\, d^4x \\ \mathcal{L}(\varphi, \partial_\mu\varphi) = g(\varphi) + h(\varphi)\, \partial_\mu\varphi\, \partial^\mu\varphi \end{cases}$

Exemplo 3: Dilatações

$$\begin{cases} x'^\mu = \lambda x^\mu \\ \phi'(\lambda x) = \underbrace{\lambda^{-\Delta} \phi(x)}_{\text{definição conforme}} \end{cases} \Rightarrow \begin{cases} \dfrac{\partial x^\nu}{\partial x'^\mu} = \lambda^{-1} \\ f(\phi(x)) = \lambda^{-\Delta} \phi(x) \end{cases} \Rightarrow S' = \int \lambda^4 \mathcal{L}(\lambda^{-\Delta}\phi, \lambda^{-\Delta-1}\partial_\mu \phi)\, d^4x$$

- Se $\phi = \varphi$ (escalar) $\Rightarrow S = \int \underbrace{\partial_\mu \varphi\, \partial^\mu \varphi}_{(?)}\, d^4x$

- Para $S = S' \Rightarrow \Delta = 1$

tilibra

Transformações infinitesimais

$$\begin{cases} x'^{\mu} \simeq x^{\mu} + \omega_a \dfrac{\delta x^{\mu}}{\delta \omega^a} \\ f(\phi) = \phi'(x') \simeq \phi(x) + \omega_a \dfrac{\delta f(\phi(x))}{\delta \omega^a} \end{cases}, \quad \text{onde "}a\text{" depende da transformação}$$

$$\Rightarrow \phi'(x') \simeq \phi(x) + \omega_a \left(\frac{\partial \phi}{\partial x^{\mu}} \frac{\delta x^{\mu}}{\delta \omega^a} \right) \simeq \phi(x') + \omega_a \frac{\delta x^{\mu}}{\delta \omega^a} \partial_{\mu} \phi(x')$$

Gerador da transformação (G_a)

$$\phi'(x) - \phi(x) \equiv -i \omega_a G_a \phi(x) = \left(\cancel{\phi(x')} + \omega_a \frac{\delta x^{\mu}}{\delta \omega^a} \partial_{\mu} \phi(x') \right) - \left(\cancel{\phi(x)} + \omega_a \frac{\delta f}{\delta \omega^a} \right)$$

$$\Leftrightarrow \boxed{i G_a \phi = \frac{\delta x^{\mu}}{\delta \omega^a} \partial_{\mu} \phi - \frac{\delta f}{\delta \omega^a}}$$

Exemplo 1: translação

$$\begin{cases} x'^{\mu} = x^{\mu} + a^{\mu} \\ x'^{\mu} = x^{\mu} + \omega_a \dfrac{\delta x^{\mu}}{\delta \omega^a} \end{cases} \xrightarrow{a \to \nu} x'^{\mu} = x^{\mu} + \omega_{\nu} \frac{\delta x^{\nu}}{\delta \omega^{\nu}} = x^{\mu} + \omega^{\mu}$$

$$\Rightarrow \begin{cases} \dfrac{\delta x^{\mu}}{\delta \omega^{\nu}} = \delta^{\mu}_{\nu} \\ \dfrac{\delta f}{\delta \omega^{\nu}} = 0 \end{cases} \Rightarrow i G_{\nu} \phi = \delta^{\mu}_{\nu} \partial_{\mu} \phi - 0 \Leftrightarrow G_{\mu} = -i \partial_{\mu} \equiv P_{\mu}$$

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Exemplo ai: Lorentz

- $x'^\mu = x^\mu + \omega^\mu{}_\nu x^\nu = x^\mu + \omega_{\rho\nu}\eta^{\rho\mu}x^\nu \Rightarrow \omega_a \frac{\delta x^\mu}{\delta\omega^a} = \omega_{\rho\nu}\eta^{\rho\mu}x^\nu$

$a \to \rho\nu$: $\omega_{\rho\nu}\frac{\delta x^\mu}{\delta\omega^{\rho\nu}} = \omega_{\rho\nu}\eta^{\rho\mu}x^\nu$

OBS: $\omega_{\rho\nu}\eta^{\rho\mu}x^\nu \underset{\nu\leftrightarrow\rho}{=} \omega_{\nu\rho}\eta^{\nu\mu}x^\rho = -\omega_{\rho\nu}\eta^{\nu\mu}x^\rho \Leftrightarrow$

$\Leftrightarrow \omega_{\rho\nu}\eta^{\rho\mu}x^\nu = \frac{1}{2}\left(\omega_{\rho\nu}\eta^{\rho\mu}x^\nu + \omega_{\rho\nu}\eta^{\rho\mu}x^\nu\right) = \frac{1}{2}\left(\omega_{\rho\nu}\eta^{\rho\mu}x^\nu - \omega_{\rho\nu}\eta^{\nu\mu}x^\rho\right)$

portanto $\omega_{\rho\nu}\frac{\delta x^\mu}{\delta\omega^{\rho\nu}} = \omega_{\rho\nu}\eta^{\rho\mu}x^\nu = \omega_{\rho\nu}\cdot\frac{1}{2}\left(\eta^{\rho\mu}x^\nu - \eta^{\nu\mu}x^\rho\right)$

$\Rightarrow \frac{\delta x^\mu}{\delta\omega^{\rho\nu}} = \frac{1}{2}\left(\eta^{\rho\mu}x^\nu - \eta^{\nu\mu}x^\rho\right)$

- $f(\phi) = L_\Lambda \phi \quad\rightarrow\quad f(\phi) \simeq \left(1 - \frac{i}{2}\omega_{\rho\nu}S^{\rho\nu}\right)\phi$ (?)

$\Rightarrow \frac{\delta f}{\delta\omega^{\rho\nu}} \simeq -\frac{i}{2} S^{\rho\nu}\phi$

- Substituindo na expressão do gerador:

$\Rightarrow \frac{i}{2}\omega_{\rho\nu} G^{\rho\nu}\phi = \frac{1}{2}\omega_{\rho\nu}\left(\eta^{\rho\mu}x^\nu - \eta^{\nu\mu}x^\rho\right)\partial_\mu\phi + \frac{i}{2}S^{\rho\nu}\phi$

$\Rightarrow G^{\rho} \equiv L^{\rho\nu} = i\left(x^\rho\partial^\nu - x^\nu\partial^\rho\right) + S^{\rho\nu}$

↳ angular

① Jacobiano das Transformações infinitesimais

- $x'^\mu = x^\mu + \omega_a \frac{\delta x^\mu}{\delta\omega_a} \Rightarrow \frac{\partial x'^\nu}{\partial x^\mu} = \delta^\nu_\mu + \partial_\mu\left(\frac{\delta x^\nu}{\delta\omega_a}\right)$

$\Rightarrow \left|\frac{\partial x'}{\partial x}\right| \simeq 1 + \partial_\mu\left(\omega_a\frac{\delta x^\mu}{\delta\omega_a}\right)$ e $\left|\frac{\partial x}{\partial x'}\right| \simeq 1 - \partial_\mu\left(\omega_a\frac{\delta x^\mu}{\delta\omega_a}\right)$

OBS: $\det(\mathbb{1} + \omega) \simeq 1 + \mathrm{Tr}(\omega)$

Substituindo em S':

$$S' = \int \left(1 + \partial_\mu\left(\omega_a \frac{\delta x^\mu}{\delta \omega_a}\right)\right) \cdot \mathcal{L}\left(\phi + \omega_a \frac{\delta f}{\delta \omega_a}, \left(\delta_\mu^\nu - \partial_\mu\left(\omega_a \frac{\delta x^\nu}{\delta \omega_a}\right)\right)\left(\partial_\nu \phi + \partial_\nu\left(\omega_a \frac{\delta f}{\delta \omega_a}\right)\right)\right)$$

fazendo $\delta S = S' - S = 0$:

$$\Rightarrow \delta S = -\int j^\mu_a \, \partial_\mu \omega_a \, d^4x \, , \qquad j^\mu_a = \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L}\right) \frac{\delta x^\nu}{\delta \omega_a} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\delta f}{\delta \omega_a}$$

corrente de Noether