Substituindo em S':

$$S' = \int \left(1 + \partial_\mu\left(\omega_a \frac{\delta x^\mu}{\delta \omega_a}\right)\right) \cdot \mathcal{L}\left(\phi + \omega_a \frac{\delta f}{\delta \omega_a}, \left(\delta_\mu^{\ \nu} - \partial_\mu\left(\omega_a \frac{\delta x^\nu}{\delta \omega_a}\right)\right)\left(\partial_\nu \phi + \partial_\nu\left(\omega_a \frac{\delta f}{\delta \omega_a}\right)\right)\right)$$

fazendo $\delta S = S' - S = 0$:

$$\Rightarrow \delta S = -\int j^\mu_{\ a}\, \partial_\mu \omega_a\, d^4x \quad , \quad j^\mu_{\ a} = \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_{\ \nu} \mathcal{L}\right) \frac{\delta x^\nu}{\delta \omega_a} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\delta f}{\delta \omega_a}$$

↳ corrente de Noether

⇒ Simetrias Contínuas → Correntes Conservadas → Carga conservada

$$Q_a \equiv \int j^0_{\ a}\, d^3x \rightarrow \frac{dQ_a}{dt} = \int \partial_0 j^0_{\ a}\, d^3x = -\int \partial_i j^i_{\ a}\, d^3x = 0$$

↳ carga

↳ termo de borda

Ambiguidade devido à liberdade da escolha

$$j^\mu_{\ a} \rightarrow j^\mu_{\ a} + \partial_\nu B^{\nu\mu}_{\ \ a}$$ tal que $\partial_\mu j^\mu_{\ a}$ não é alterado pois

↳ antissimétrico

$$\underbrace{\partial_\mu \partial_\nu}_{} B^{\nu\mu}_{\ \ a} = 0$$

↳ simétrico

Tensor Energia Momento

$$\begin{cases} x^\mu \rightarrow x^\mu + \varepsilon^\mu \quad (a \rightarrow \mu) \\ f = 1 \end{cases} \Rightarrow \frac{\delta x^\mu}{\delta \varepsilon^\nu} = \delta^\mu_\nu \quad e \quad \frac{\delta f}{\delta \varepsilon^\nu} = 0$$

$$\Rightarrow T_c^{\ \mu\nu} = -\eta^{\mu\nu}\mathcal{L} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi \quad , \quad \partial_\mu T^{\mu\nu} = 0$$

OBS: $P^\nu = \int T^{0\nu} d^3x \Rightarrow P^0 = \int T^{00} d^3x = \int \left(\frac{\partial \mathcal{L}}{\partial \dot\phi}\dot\phi - \mathcal{L}\right) d^3x$

densidade Hamiltoniana

Construção de Belinfante

- $T^{\mu\nu}_{B} = T^{\mu\nu}_{C} + \partial_\rho B^{\rho\mu\nu}$, $B^{\rho\mu\nu} = - B^{\mu\rho\nu}$ e $T^{\mu\nu}_{B} = T^{\nu\mu}_{B}$

- Lorentz
$$\frac{\delta x^\rho}{\delta \omega_{\mu\nu}} = \frac{1}{2}\left(\eta^{\rho\mu} x^\nu - \eta^{\rho\nu} x^\mu\right)$$
$$\frac{\delta f}{\delta \omega_{\mu\nu}} = -\frac{i}{2} S^{\mu\nu}\phi$$

- $$j^{\mu\nu\rho} = T^{\mu\nu}_{C} x^\rho - T^{\mu\rho}_{C} x^\nu + \frac{i}{2} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} S^{\nu\rho}\phi$$

- Escolhendo : $$B^{\mu\rho\nu} = \frac{i}{4}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} S^{\nu\rho}\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\rho \phi)} S^{\mu\nu}\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} S^{\mu\rho}\phi \right)$$

verifica-se que: $$j^{\mu\nu\rho} = T^{\mu\nu}_{B} x^\rho - T^{\mu\rho}_{B} x^\nu$$

Exemplo

Eletromagnetismo :

$$\begin{cases} \mathcal{L} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \\ T^{\mu\nu}_{c} = -\eta^{\mu\nu}\mathcal{L} + F^{\mu\alpha}\partial^\nu A_\alpha \end{cases} , \quad \text{se} \quad B^{\alpha\mu\nu} = F^{\alpha\mu} A^\nu$$

$$\Rightarrow T^{\mu\nu}_{B} = T^{\mu\nu}_{c} + F^{\alpha\mu}\partial_\alpha A^\nu + \partial_\alpha F^{\alpha\mu} A^\nu$$ "Simétrico"

Tensor Energia - Momento para o Eletromagnetismo

$$\tilde{T}^{\mu\nu} = F^{\mu\alpha} F^{\nu}{}_{\alpha} - \frac{1}{4}\eta^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} = T^{\mu\nu}_{B} - \left(\partial_\alpha F^{\alpha\mu}\right) A^\nu$$

↳ "on shell" = 0

2) Método Alternativo

- Promover a métrica do espaço-tempo $\eta_{\mu\nu} \to g_{\mu\nu}$ "curvo"
- Determinar $\dfrac{\delta S}{\delta g_{\mu\nu}}$
- Substituir $g_{\mu\nu} \equiv \eta_{\mu\nu}$

onde $g'_{\mu\nu} = \dfrac{\partial x^\alpha}{\partial x'^\mu} \dfrac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}$

1) Translação

- $x'^\mu = x^\mu + \varepsilon^\mu \Rightarrow \dfrac{\partial x^\alpha}{\partial x'^\mu} = \dfrac{\partial}{\partial x'^\mu} \cdot (\partial x'^\alpha - \varepsilon^\alpha) = \dfrac{\partial x'^\alpha}{\partial x'^\mu} - \dfrac{\partial \varepsilon^\alpha}{\partial' x^\mu}$

$$= \delta^\alpha_\mu - \partial'_\mu \varepsilon^\alpha \simeq \delta^\alpha_\mu - \partial_\mu \varepsilon^\alpha$$

$$\Rightarrow g'_{\mu\nu} = (\delta^\alpha_\mu - \partial_\mu \varepsilon^\alpha)(\delta^\beta_\nu - \partial_\nu \varepsilon^\beta)\, g_{\alpha\beta}$$

$$g'_{\mu\nu} = g_{\mu\nu} - \underbrace{(\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu)}_{\hookrightarrow\ \delta g_{\mu\nu}}$$

2) Eletromagnetismo em Espaço Curvo

- $S = \dfrac{1}{4} \displaystyle\int d^4x\, F_{\mu\nu} F^{\mu\nu} = \dfrac{1}{4} \int d^4x \sqrt{-\det(\eta)}\, F_{\mu\nu} F_{\rho\sigma} \eta^{\mu\rho} \eta^{\nu\sigma}$

$$\Rightarrow S_{curvo} = \frac{1}{4} \int d^4x \sqrt{-g}\, F_{\mu\nu} F_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma}$$

3) Campo Escalar livre de massa m

- $S = \displaystyle\int d^4x \sqrt{-g}\, (g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + m^2 \varphi^2)$

- $\dfrac{\delta S}{\delta g_{\mu\nu}} \to T^{\mu\nu} = -g^{\mu\nu} \mathcal{L} + \partial^\mu \varphi \partial^\nu \varphi \to -\eta^{\mu\nu} \mathcal{L} + \partial^\mu \varphi \partial^\nu \varphi$

/ /

Ondas No Vácuo

$$\begin{cases} \vec{\nabla}\cdot\vec{E} = 0 \\ \vec{\nabla}\cdot\vec{B} = 0 \end{cases} \quad e \quad \vec{\nabla}\times\vec{B} = \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} \quad ; \quad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

1) $\frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}\right) = \frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B}) \Rightarrow \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = \vec{\nabla}\times\frac{\partial\vec{B}}{\partial t}$

$\Rightarrow \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = \vec{\nabla}\times(-\vec{\nabla}\times\vec{E}) \Leftrightarrow \boxed{\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = \nabla^2\vec{E}}$

2) $\frac{\partial}{\partial t}\left(-\frac{\partial\vec{B}}{\partial t}\right) = \frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E}) \Rightarrow -\frac{\partial^2\vec{B}}{\partial t^2} = \vec{\nabla}\times\frac{\partial\vec{E}}{\partial t}$

$\Rightarrow -\frac{\partial^2\vec{B}}{\partial t^2} = \vec{\nabla}\times(c^2\,\vec{\nabla}\times\vec{B}) \Leftrightarrow \boxed{\frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2} = \nabla^2\vec{B}}$

Exemplo

- $\vec{E} = (0, E(t,x), 0)$ onde $E(t,x) = f(x-ct) + g(x+ct)$

$\Rightarrow E(t,x) = E_0 \operatorname{sen}(kx - \omega t)$, $k = \frac{\omega}{c}$

- $\frac{\partial\vec{B}}{\partial t} = -\vec{\nabla}\times\vec{E} \longrightarrow \vec{B} = (0, 0, B(t,x))$

$\Rightarrow \frac{\partial B(t,x)}{\partial t} = -\frac{\partial E}{\partial x} = -kE_0\cos(kx-\omega t) = -\frac{E_0}{c}\,\omega\cos(kx-\omega t)$

$\Rightarrow B(t,x) = \frac{E_0}{c}\operatorname{sen}(kx-\omega t)$