Eletromagnetismo 2 – Lista 4

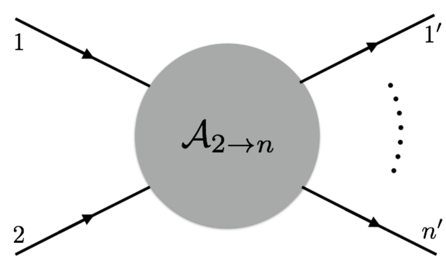
O campo escalar, III

Let's consider a theory of a real scalar field in two dimensions (1 time + 1 space), with Lagrangian density given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \sum_{n=3}^{\infty}\frac{\lambda_n}{n!}\phi^n\,.$$

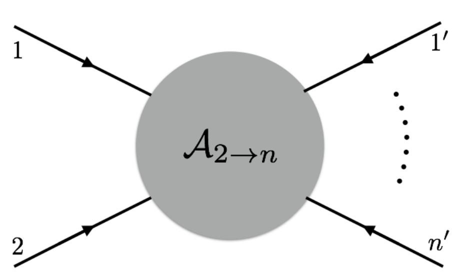
What is the scaling dimension of ϕ in two spacetime dimensions? You can deduce this from $S = \int d^2x\mathcal{L}$, remembering that S has to be dimensionless in units of $\hbar = 1$. What does this imply about the nature of the coupling constants λ_n? Are they relevant, irrelevant or marginal?

Let's consider now the scattering amplitudes $\mathcal{A}_{2\to n}$ for the tree-level processes in which 2 initial particles go into $n \geq 2$ final particles, like in this figure:



The arrows in the lines are meant to show the direction in which the momentum flows, and do not distinguish between particles and antiparticles like in the scalar Yukawa theory seen in class (sorry for the change of notation!).

One can adopt the useful convention that all particles are taken to be incoming, with the understanding that in the end all but two particles are crossed to be outgoing. This means that all the momenta are taken to be incoming, like this:



It is convenient to go to light-cone coordinates and write the momentum of the $i-$th particle as $(p_i^+, p_i^-) = (p_i^0 + p_i^1, p_i^0 - p_i^1)$ in terms of a single real parameter a_i as

$$p_i = (ma_i, m/a_i)\,.$$

This guarantess the mass-shell condition $p_i^+ p_i^- = m^2$. Understand this point (you need to understand how the Minkowski metric $\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is rewritten in light-cone coordinates). From now on, we can set $m = 1$ to simplify expressions.

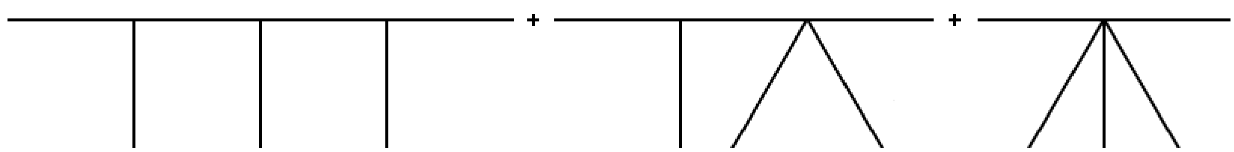
How do write energy and momentum conservation in terms of the a_i's?

Convince yourself that the propagator for a particle carrying momentum $p = \sum_i p_i$ is given by

$$G(p) = \frac{1}{p^2 - 1} = \frac{1}{\left(\sum_i a_i\right)\left(\sum_j 1/a_j\right) - 1}\,.$$

Remember we have set $m = 1$.

The scope of this exercise is now to check whether we can choose the coupling λ_n appropriately and cancel all processes of particle production. In other words, we want $\mathcal{A}_{2\to n\geq 3} = 0$. For example, the (tree-level) processes contributing to $\mathcal{A}_{2\to 3}$ are given by the diagrams in this figure:



The first diagram contains λ_3 only, the second diagram λ_3 and λ_4, and the last diagram is essentially given by λ_5, namely it is momentum independent. Check that you can fix λ_4 in terms of λ_3 in a way that the sum of the first two diagrams becomes constant. You can now eliminate these two diagrams by fixing λ_5 to be minus that constant. What are λ_4 and λ_5?

Now we want to generalize this to $n > 3$ and find what are the other couplings $\lambda_{n\geq 6}$ in terms of λ_3. This can be done recursively, using a clever choice of momenta.

Let's change notation a bit, remembering of our convention of taking all particles to be incoming. We consider an amplitude of n particles: $n = 4$ would be $2 \to 2$ scattering, $n = 5$ would be $2 \to 3$ scattering and so on. We want then to cancel all amplitudes with $n \geq 5$.

All tree-level diagrams for n particles, except for the constant one equal to λ_n, can be factorized in a left blob and a right blob connected by a propagator

$$G_{L\to R} = \frac{1}{\left(\sum_{i\in L} a_i\right)\left(\sum_{j\in L} 1/a_j\right) - 1}\,. \tag{1}$$

This allows to pick a convenient choice of momenta which simplifies the evaluation of the recursive relations among the λ_n's. This convenient choice turns out to be

$$a = \{a_1(x), 1, x, x^2, \ldots, x^{n-3}, a_n(x)\}\,, \tag{2}$$

with $a_i(x)$ and $a_n(x)$ being determined by the conservation of energy and momentum you've written down above. In the limit of $x \to \infty$,

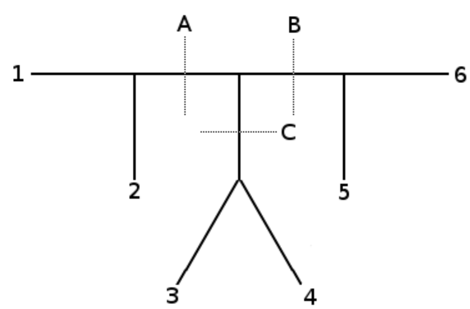
$$a_1(x) = -1 + \mathcal{O}(1/x)\,, \qquad a_n(x) = -x^{n-3}\left(1 + \mathcal{O}(1/x)\right)\,, \tag{3}$$

and one can check that

$$G_{L\to R} = \begin{cases} -1, & \text{if } a_j^L = (a_1, a_2, \ldots, a_k), a_j^R = (a_{k+1}, a_{k+2}, \ldots, a_n) \\ 0, & \text{otherwise} \end{cases}\,,$$

that is, only the diagrams where the particles are ordered contribute. To see this, it is sufficient to evaluate a few cases. For example, the set $a_j^L = \{a_2, a_3\}$ gives a vanishing propagator and it is easy to see that any other set of two or more momenta not including a_1 goes to zero as well. Similarly, the set $a_j^L = \{a_1, a_{i\geq 3}\}$ also gives zero. On the other hand, the set $a_j^L = \{a_1, a_2\}$ yields -1. By adding $a_{i\geq 4}$ to this set, we get zero again. The only case left to analyze is if an ordered set of any size continues to converge to -1, which it does.

An important consequence is that only ordered line-type diagrams survive, as shown pictorially in this figure for the case of six particles:



This diagram seems to be non-vanishing since it is ordered for the cuts A and B. However, for the cut C it is $a_j^L = \{a_3, a_4\}$, which implies that this diagram does in fact vanish. In general, only (ordered) line-type diagrams survive the large x limit.

After these considerations, one is ready to compute the amplitude $\mathcal{A}_{2\to n-2}$. It is useful to start thinking about the diagram that has λ_{n-k} as its rightmost vertex. This diagram factorizes into the form $\mathcal{A}_{2\to k} \cdot \lambda_{n-k}$. Check that imposing $\mathcal{A}_{2\to k}$ to vanish, one gets the recursion relation

$$\mathcal{A}_{2\to n-2} = -\lambda_3\lambda_{n-1} + (\lambda_3)^2\lambda_{n-2} - \lambda_4\lambda_{n-2} + \lambda_n = 0\,.$$

This can be solved by setting $\lambda_n = \gamma^n$, writing a generic combination of the two roots, and imposing consistency with the expression for λ_4 in terms of λ_3 that you've derived above.

After you've obtained the expression for a generic λ_n in terms of λ_3, plug this in the potential in $\mathcal{L}$ and resum. You should end up with

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{6\lambda_3^2}\left(2e^{-\lambda_3\phi} + e^{2\lambda_3\phi} - 3\right) .$$

This is called Bullough-Dodd model and it is a famous example of an *integrable model.* One of the characteristics of integrable models is that they do not produce particles. In a sense, we have rediscovered the existence of this model from first principles, by imposing absence of particle production in the scattering processes.

Repeat the same kind of analysis for the case of a real scalar field, like in $\mathcal{L}$ above, but with a $\mathbb{Z}_2$ symmetry: the Lagrangian now has to be invariant under $\phi \to -\phi$. You should find that the final Lagrangian is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{\lambda_4}(\cosh(\sqrt{\lambda_4}\phi) - 1) .$$

This is called the cosh-Gordon model and is another famous example of integrable theory in two dimensions.

You can read more about this subject of integrable models and exact S-matrices in the beatiful review by P. Dorey:
`https://arxiv.org/pdf/hep-th/9810026.pdf`.