Buracos negros

1. Demonstre que a métrica de Schwarzschild pode ser colocada em forma isotrópica

$$ds^2 = \left(\frac{1 - \frac{GM}{2\rho}}{1 + \frac{GM}{2\rho}} \right)^2 dt^2 - \left(1 + \frac{GM}{2\rho} \right)^4 \left(d\rho^2 + \rho^2 d\Omega_2^2 \right) .$$

   Onde é o horizonte nestas novas coordenadas?

2. **Órbitas no fundo de Schwarzschild**

   (a) Derive as equações do movimento para partículas com massa e sem massa que se propagam em um fundo de Schwarzschild.

   (b) Usando as simetrias do problema, escreva a equação radial na forma de uma equação de tipo Newtoniano com um potencial "efetivo"

$$\frac{1}{2} \left(\frac{dr}{d\lambda} \right)^2 + V(r) = E \,,$$

   onde λ é um parâmetro apropriado. Qual é a diferência entre o caso Newtoniano e a RG?

   (c) Descreva as possíveis órbitas.

   (d) Usando Mathematica, faça plots das órbitas de partículas com e sem massa, para quatro ou cinco valores diferentes do momento angular.

   (e) Ache as condições de estabilidade das órbitas.

3. Um observador cai radialmente dentro de um buraco negro esférico de massa M. Ele começa a cair partindo de uma posição de repouso a $r = 10M$ (em unidades naturais) relativamente a um outro observador fixo. Quanto tempo passa no relógio do observador que cai, antes de atingir a singularidade?

4. Um observador que cai em um buraco negro esférico pode receber informações sobre eventos que acontecem fora do buraco negro? Há uma região fora do buraco negro que o observador dentro do buraco negro não pode ver?

5. A Natureza não apresenta massas negativas, mas *just for fun* vamos considerar uma geometria de Schwarzschild com $M < 0$. Estude o comportamento de raios de luz radiais nesta geometria e desenhe o correspondente diagrama de Eddington-Finkelstein. Esta geometria representa um buraco negro?

6. **(Extremal) RN black holes**

To give a black hole a charge we need to couple gravity with an electromagnetic field. This coupling, in 4 dimensions, is given by the *Einstein-Maxwell action*[1]

$$S_{EM} = \frac{1}{16\pi G}\int d^4x\,\sqrt{-g}\left(R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right)\,, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,.$$

a) Write the Einstein equations of motion $\delta S_{EM}/\delta g^{\mu\nu} = 0$ resulting from this action and the Maxwell equations of motion $\delta S_{EM}/\delta A_\mu = 0$.

b) Just by looking at the equations of motion compute the scalar curvature R.

c) Consider now the following Ansatz for the metric and gauge field:

$$\begin{aligned} ds^2 &= -\frac{1}{H(r)^2}dt^2 + H(r)^2\left(dr^2 + r^2 d\Omega_2^2\right)\,, \\ F_{tr} &= -F_{rt} = \alpha\frac{1}{H(r)^2}\frac{dH(r)}{dr}\,. \end{aligned}$$

Here α is some constant to be determined. All other components of $F_{\mu\nu}$ are zero. In particular, this means that the gauge field we are considering is an electric field.

Plug the Ansatz above in the equations of motion (you may use `Mathematica`) and find the correct value of the constant α and the equation that $H(r)$ must satisfy in order for the Ansatz to be a solution of the equations of motion.

d) What is the general solution of the equation for $H(r)$ that you have found? For this part of the problem it might be easier (although this is not necessary) to transform the $dr^2 + r^2 d\Omega_2^2$ part of the metric to Cartesian coordinates.

e) In the most general solution there are two integration constants (because the equation is second order). Fix one of them by requiring asymptotic flatness (that is, at $r \to \infty$ the metric should be Minkowski). The second integration constant should be fixed by requiring that the black hole has charge Q. This is done by computing the flux threading the S^2 (at fixed r) of the dual field strength:[2]

$$Q \equiv \int_{S^2} \star F_{\theta\phi}\, d\theta d\phi\,, \tag{1}$$

[1] Here I am appropriately rescaling the gauge field A_μ so that the $1/(16\pi G)$ factor becomes an overall factor.

[2] You can understand this by writing Maxwell's equations using differential forms (we'll do this later, maybe). One has

$$dF = \star J_m\,, \qquad d\star F = \star J_e\,,$$

where J_e and J_m are the electric and magnetic currents (if we include not-yet-observed magnetic monopoles in the equations). Integrating over a ball around the electric and magnetic charges and applying Stokes, you get

$$e = \int_{S^2}\star F\,, \qquad m = \int_{S^2} F\,.$$

where $\star F_{\mu\nu}$ is the Hodge dual of $F_{\mu\nu}$. This is oriented along the S^2 (unlike $F_{\mu\nu}$ which in our Ansatz has legs along the t and r directions, but not along the S^2 directions). The explicit definition of the Hodge dual of a form $F_{\mu\nu}$ is

$$\star F_{\rho\sigma} = \frac{1}{2}\sqrt{-g}\,\epsilon_{\rho\sigma}{}^{\mu\nu}F_{\mu\nu}\,,$$

where $\epsilon_{\rho\sigma\mu\nu}$ is the totally antisymmetric tensor ($\epsilon_{0123} = +1$, with even permutation of the indices being equal to $+1$, odd permutations being equal to -1, and repeated indices being equal to 0.).

f) Can you superimpose different solutions? What does this represent? Ask me why this happens! (It is a "miracle" called supersymmetry...).

g) Take the so-called *near-horizon limit*, $r \to 0$. Show that the resulting metric is $AdS_2 \times S^2$. Write down the relation between the radius of curvature of the space L (both AdS_2 and S^2 have the same radius L) and the charge Q.

h) Do you know how to add a magnetic field? First of all, write down how you would modify the Ansatz for $F_{\mu\nu}$ and then try to solve the resulting equations.

7. **Generic RN black holes**

What we have seen in the previous exercise is in fact a very special case of RN black hole, the case in which the mass M and the charge Q of the black hole are the same (this is why it was called "extremal").

Consider the more general RN black hole

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1}dr^2 + r^2 d\Omega_2^2\,, \\ F_{tr} &= \alpha\frac{Q}{r^2}\,. \end{aligned}$$

a) Using `Mathematica` check that this is a solution to the equations of motion that you have obtained in the previous problem. In particular determine what α is.

b) Find the position of the horizons $r_\pm$. What do we need to assume about M and Q? Compute the temperature of the black hole (using the outer horizon $r = r_+$).[3] What happens when $M = Q$?

c) Compute the appropriate curvature invariant to understand the nature of the singularities at $r = r_\pm$ and at $r = 0$.

8. **AdS/Schwarzschild**

Let's have another look at

$$ds^2 = -\left(1 + \frac{r^2}{L^2} - \frac{r_0^4}{L^2r^2}\right)dt^2 + \left(1 + \frac{r^2}{L^2} - \frac{r_0^4}{L^2r^2}\right)^{-1}dr^2 + r^2 d\Omega_3^2\,,$$

[3]Look at the *Lightning review of GR* to do this.

the *AdS/Schwarzschild black hole* in global coordinates.

Find the position of the horizon of this black hole (which we'll call r_H) and compute its temperature (this is obtained by requiring no conical singularities for the Euclideanized metric, as explained in the notes *Lightning review of GR*). Plot $1/T$, the inverse temperature, on the vertical axis versus the position of the horizon r_H on the horizontal axis. Discuss this plot. In particular, can you understand why people talk about "large" and "small" AdS black holes?

9. Considere a métrica de Schwarzschild nas coordenadas de Schwarzschild (t, r, θ, ϕ). Faça explicitamente todas as transformações de coordenadas para chegar nas formas da métrica de Eddington-Finkelstein e de Kruskal-Szekeres.

10. Considere a métrica

$$ds^2 = -\left(1 - \frac{M}{r}\right)^2 + \left(1 - \frac{M}{r}\right)^{-2} dr^2 + r^2 d\Omega_2^2 \,.$$

a) Faça uma transformação para novas coordenadas de tipo Eddington-Finkelstein, tais que $g_{rr} = 0$, e mostre que a geometria é não singular em $r = M$.

b) Estude o comportamento do cone de luz desta geometria. É um buraco negro?

11. Depois de ter cruzado o horizonte de eventos de um buraco negro, qual é o tempo próprio mais longo que o observador pode experimentar antes de atingir a singularidade?

12. Demonstre que em um diagrama de Kruskal $\left|\frac{dV}{dU}\right| > 1$ para a linha de mundo de tipo tempo de uma partícula.

13. Alice e Bob se encontram a uma distância R de um buraco negro de Schwarzschild de massa M, onde R é tal que

$$\sqrt{\frac{R}{2M} - 1}\, e^{R/4M} = \frac{1}{2} \,.$$

A posição angular deles é constante. Bob deixa esta posição a $t = 0$ e viaja através do horizonte de eventos ao longo de uma linha reta em um diagrama de Kruskal, até ser destruido pela singularidade, no ponto que a singularidade cruza a linha $U = 0$. Alice continua na posição original.

a) Desenhe as linhas de mundo de Alice e Bob em um digrama de Kruskal.

b) A linha de mundo de Bob é de tipo tempo?

c) Qual é o tempo máximo na coordenada t de Schwarzschild depois de Bob ter saido da posição original, que Alice pode enviar um sinal de luz para Bob, antes que ele seja destruido pela singularidade?