24/12/2016

$T^{ij} \rightarrow D^2$ componentes
$i,j = 1 \ldots D$

$$T^{ij} = \begin{cases} A^{ij} = T^{ij} - T^{ji} \rightarrow \frac{D(D-1)}{2} \\ S = S^{ii} \left(= \sum_{i=1}^{D} S^{ii}\right) \rightarrow 1 \\ \tilde{S}^{ij} = \underbrace{\left(T^{ij} + T^{ji}\right)}_{\equiv S^{ij}} - \frac{\delta_{ij} S}{D} \rightarrow \frac{D(D+1)}{2} - 1 \end{cases}$$

$$\sum_{i=1}^{D} \tilde{S}^{ii} = 0$$

$\tilde{S}^{ij}$ é sem traço:

$$\sum_{i,j} \delta^{ij} \tilde{S}^{ij} = \sum_{i} \tilde{S}^{ii} = \tilde{S}^{ii}$$

$$\sum_{i,j} \delta^{ij} \left(S^{ij} - \frac{\delta^{ij}}{D} S\right) = S^{ii} - \frac{\delta^{ij}\delta^{ii}}{D} S = S - \frac{D}{D} S = 0$$

$$D^2 = \frac{D(D-1)}{2} \oplus 1 \oplus \left(\frac{D(D+1)}{2} - 1\right)$$

$D = 3 \rightarrow 9 = 3 \oplus 1 \oplus 5$

$$D(R) = \begin{pmatrix} 3\times 3 & & \\ & 1\times 1 & \\ & & 3\times 5 \end{pmatrix}_{9\times 9}$$

Subgrupos

$SO(2)$ $SO(3)$

$$\underbrace{\begin{array}{c} SO(2) \\ \begin{array}{cccc} 1 & 2 & 3 & \\ \left(\begin{array}{ccc} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{array}\right) & & & \begin{array}{cc} x & 1 \\ y & 2 \\ z & 3 \end{array} \end{array} \end{array}}_{SO(3)}$$

$V^i \quad i = 1, 2, 3$

$\underbrace{(V^1, V^2)}_{\text{Vetor sob } SO(2)}$ V^3 $\rightarrow$ Não se transforma sob $SO(2)$

T^{ij}

$\underset{A^{ij}}{3} = \overset{\text{dubleto}}{2} \oplus 1$

singletos = partes de T^{ij} que não se transf. sob $SO(2)$

$\underset{\tilde{S}^{ij}}{5} = \underbrace{2 \oplus 2}_{\text{dois dubletos}} \oplus 1$

$\tilde{S}^{ij}$: $\tilde{S}^{11}, \tilde{S}^{12}, \tilde{S}^{13}, \tilde{S}^{22}, \tilde{S}^{33}$

$SO(2)$ $\quad (\tilde{S}^{11} + \tilde{S}^{22} + \tilde{S}^{33} = 0$

$\downarrow$ $\quad \longrightarrow \tilde{S}^{33}$ não é indep.)

$$\begin{cases} V^3 \to (V^3)' = V^3 \\ \Rightarrow \begin{cases} R^{33} = 1 \\ R^{13} = R^{23} = R^{31} = R^{32} = 0 \end{cases} \end{cases}$$

$\tilde{S}^{11} + \tilde{S}^{22} \left(= -\tilde{S}^{33}\right)$ não se transf. sob $SO(2)$

é o singleto. (HW)

O dois dubletos são:

$(\tilde{S}^{13}, \tilde{S}^{23})$

$(\tilde{S}^{12}, \tilde{S}^{11} - \tilde{S}^{22})$

Tensores invariantes

δ^{ij} $\quad \epsilon^{i_1 i_2 \dots i_D}$

(ex: $D=2$ $\epsilon^{12} = -\epsilon^{21} = 1$

$D=3$ $\epsilon^{123} = \epsilon^{231} = \dots = 1$

$\vdots$)

*) δ_{ij}

$$R^T R = \mathbb{1} = R R^T$$

$$\delta^{ij} \to (\delta^{ij})' = R^{ik} R^{jk} \delta^{kl}$$

$$= R^{ik} R^{jk} = R^{jk} R^{ik} = R^{jk} (R^T)^{ki}$$

$$= \delta^{ji} = \delta^{ij}$$

*) $\underbrace{\epsilon^{ijk\dots n}}_{D\text{-indices}}$

$(\det R = 1)$ $\epsilon^{pqr\dots s}$

$$\epsilon^{pqr\dots s} \quad \underbrace{\det R}_{\epsilon^{ijk\dots l} R^{i1} R^{j2} \dots R^{l0}} = \epsilon^{pqr\dots s}$$

$$\Rightarrow \epsilon^{ijk\dots l} R^{ip} R^{jq} \dots R^{ls}$$

$$D = 2 \quad \begin{vmatrix} R^{11} & R^{12} \\ R^{21} & R^{22} \end{vmatrix} = R^{11} R^{22} - R^{12} \cdot R^{21}$$

$$\epsilon^{ij} R^{i1} R^{j2} = \epsilon^{12} R^{11} R^{22} + \underbrace{\epsilon^{21}}_{=-1} R^{21} R^{12}$$

$D = 2$

$$\epsilon^{ij} = R^{ip} R^{jq} = \epsilon^{pq}$$

$$\left.\begin{matrix} p = 1 \\ q = 1 \end{matrix}\right\} \epsilon^{ij} R^{i1} R^{j1} = 0$$

$$\underbrace{\epsilon^{12}}_{=1} R^{11} R^{21} + \underbrace{\epsilon^{21}}_{=-1} R^{21} R^{11} = 0$$

$$(HW) \to \begin{matrix} p = 1 \\ q = 2 \end{matrix}$$

⊛ $\sum_k \epsilon^{ijk}\epsilon^{lnk} \to \delta^{il}\delta^{jn} - \delta^{in}\delta^{jl}$

↳ explica as relações $\vec{A}\cdot(\vec{B}\times\vec{C}) = \dots$

⊛ $\delta^{ij}A^{ij} = 0$

$$\delta^{ji}A^{ji} = \delta^{ij}(-A^{ij})$$

A^{ij} $\quad T^{ji}$

$$\| \quad \frac{T_A^{ij} + T_S^{ij}}{2}$$

seleciona a parte antissimétrica

$$(\vec{B}\times\vec{C})_i = \epsilon_{ijk} B_j C_k$$

$$(\vec{D}\times\vec{E})_i = \epsilon_{ilm} D_l E_m$$

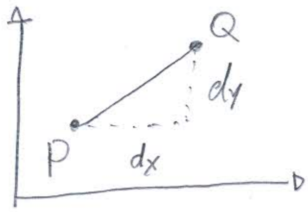
$$(\vec{B}\times\vec{C})\cdot(\vec{D}\times\vec{E}) = \\ = \epsilon_{ijk} B_j C_k \epsilon_{ilm} D_l E_m$$

Troca de coordenadas

Espaço Euclideano 2D

$P = (x, y)$

$Q = (x+dx, y+dy)$



$$ds^2 = dx^2 + dy^2$$

distância

Coord. polares

(r, θ)

$(x,y) \to (r,\theta)$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \to \begin{cases} dx = \cos\theta\, dr - r\sin\theta\, d\theta \\ dy = \sin\theta\, dr + r\cos\theta\, d\theta \end{cases}$$

$$ds^2 = dx^2 + dy^2 = dr^2 + r^2 d\theta^2$$

Coord. genéricas

$$(x,y) \longmapsto (u,v) \qquad \begin{cases} x = f(u,v) \\ y = g(u,v) \end{cases}$$

$$\begin{cases} dx = f_u\, du + f_v\, dv \\ dy = g_u\, du + g_v\, dv \end{cases}$$

$$dx^2 + dy^2 = (f_u^2 + g_u^2)\, du^2 + (f_v^2 + g_v^2)\, dv^2$$
$$+ 2(f_u f_v + g_u g_v)\, du\, dv$$

3D $(x,y,z) \longmapsto (r, \theta, \varphi)$

$$\begin{cases} x = r \sin\theta \cos\varphi \\ y = r \sin\theta \sin\varphi \\ z = r\cos\theta \end{cases} \rightarrow \begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 \\ &= dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2 \end{aligned}$$

Para conveniência futura, vamos nos acostumar, com índices gregos:

Coord. Cartesianas

$$x^\mu = (x^1, x^2, \ldots, x^D) \quad D\text{ - dimensões}$$

$$ds^2 = \boxed{\sum_{\mu=1}^{D} (dx^\mu)^2} = dx^1 dx^1 + \ldots + dx^D dx^D$$

$$= \sum_{\mu=1}^{D} \sum_{\nu=1}^{D} g_{\mu\nu}\, dx^\mu dx^\nu \qquad = \sum_{\mu=1}^{D} dx^\mu dx^\mu = (dx^\mu)^2$$

$$= g_{\mu\nu}\, dx^\mu dx^\nu$$

$$dx^\mu dx^\nu = dx^\nu dx^\mu$$

$$g_{\mu\nu} = g_{\nu\mu}$$

Aqui não faz diferença dx^μ ou dx_μ,

mas vamos nos acostumar a escrever índices contraídos um em cima e o outro em baixo.

3D Coord. esféricas

ds^2 = elemento de linha

Métrica

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & & & \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & & 1 \end{pmatrix} = \delta_{\mu\nu}$$

$D \times D$

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2$$

$$x^\mu = (r, \theta, \varphi)$$

$$x^r, x^\theta, x^\varphi$$

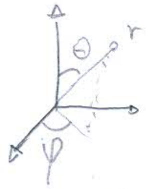
$$ds^2 = g_{\mu\nu} \, dx^\mu dx^\nu \qquad dx^\mu = (dr, d\theta, d\varphi)$$

$$\begin{cases} g_{rr} = 1 \\ g_{\theta\theta} = r^2 \\ g_{\varphi\varphi} = r^2 \sin^2\theta \end{cases} \qquad g_{r\theta} = g_{r\varphi} = 0 \qquad g_{\theta r} = g_{\varphi r} = 0$$

$$g_{\theta\varphi} = 0 \qquad g_{\varphi\theta} = 0$$

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

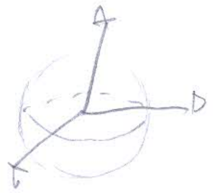
$r = a$



S^2 está embutida em $\mathbb{R}^3$ .....

2- esfera

S^2



$\theta \in [0, \pi]$

$\varphi \in [0, 2\pi]$

$$ds^2 = 0 + a^2 d\theta^2 + a^2 \sin^2\theta \, d\varphi^2$$
$$= a^2 (d\theta^2 + \sin^2\theta \, d\varphi^2)$$

$$\left. \begin{aligned} g_{\mu\nu} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ g_{\mu\nu} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \end{aligned} \right\} \mathbb{R}^3$$

Dado um espaço com métrica $g_{\mu\nu}(x)$; como posso saber se o espaço é plano (como $\mathbb{R}^3$) ou curvo (como S^2)?

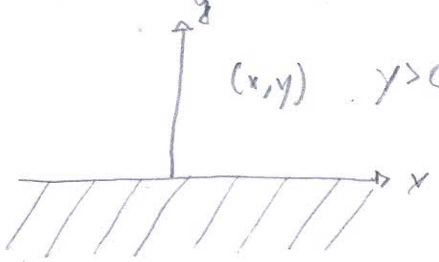
$$ds^2 = (1+u^2)\,du^2 + (1+4v^2)\,dv^2 + 2(2v-u)\,du\,dv$$

$$ds^2 = (1+u^2)\,du^2 + (1+2v^2)\,dv^2 + 2(2v-u)\,du\,dv$$

$\rightarrow$ $\boxed{\text{Curvatura}}$

$\rightarrow$ Gauss/Riemann

Exemplo: Semi-plano de Poincaré



$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

$\downarrow$

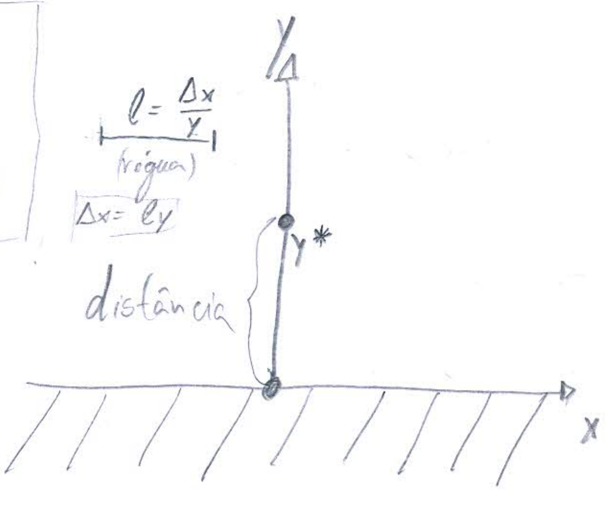
invariantes sob translações em x

$$\left.\begin{aligned} x_1 &\mapsto \tilde{x}_1 = x_1 + \Delta x \\ x_2 &\mapsto \tilde{x}_2 = x_2 + \Delta x \end{aligned}\right]\quad dx = x_1 - x_2 = \tilde{x}_1 - \tilde{x}_2$$

Mas não é invariante sob $y \mapsto y + \Delta y$

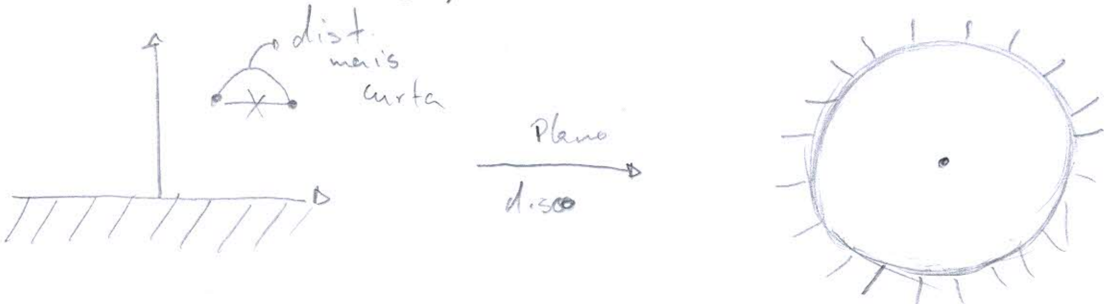
$$\boxed{\begin{aligned} g_{\mu\nu}(x+a, y) &= g_{\mu\nu}(x,y) \\ g_{\mu\nu}(x, y+a) &\neq g_{\mu\nu}(x,y) \end{aligned}}$$

$y \to 0 \Rightarrow \Delta x \to 0$



$$\text{distância} = \int ds = \int_{0^+}^{y^*} \frac{dy}{y} = \log\left(\frac{y^*}{0^+}\right) \xrightarrow[0^+ \to 0]{} +\infty$$

Escrer



disco de Poincaré

Espaço hiperbólico

Exemplo: Espaço plano em coord. de Boyer-Lindquist

($\to$ Kerr)

$$\begin{cases} x = f(r) \sin\theta \cos\varphi \\ y = f(r) \sin\theta \sin\varphi \\ z = r \cdot \cos\theta \end{cases} \to ds^2 = \underbrace{dx^2 + dy^2 + dz^2}_{\text{plano } \mathbb{R}^3}$$

$$f' = \frac{df}{dr}$$

$$(f'^2 \sin^2\theta + \cos^2\theta)\, dr^2 + (f'^2 \cos^2\theta + r^2 \sin^2\theta)\, d\theta^2$$

$$+ f^2 \sin^2\theta\, d\varphi^2 + 2\underbrace{(ff' - r)}_{=0} \sin\theta \cos\theta\, dr\, d\theta$$

$$ff' = r \to df\, f = r\, dr$$

$$\boxed{f^2 = r^2 + \text{const} = r^2 + a^2}$$

($a = 0 \to$ coord esféricas

$r = \text{cte} \to ds^2 =$ elipsoide
(não uma esfera)

$r = 0 \Rightarrow$ não é um ponto, mas um disco de raio a

$$ds^2 = a^2 \cos^2\theta\, d\theta^2 + a^2 \sin^2\theta\, d\varphi^2$$