29/12/2016

Obs: Vetores de Killing | Formalismo

isometria = transf. que deixa a métrica invariante

Transf. gerais de coordenadas

$X^{\mu} \to X'^{\mu}$

ex: $(x, y) \longrightarrow (r, \theta)$

$x = r\cos\theta$

$y = r\sin\theta$

não-lineares

mas as transf. infinitesimais são lineares

$dx = \cos\theta\, dr - \sin\theta\, r\, d\theta$

$dy = \sin\theta\, dr + \cos\theta\, r\, d\theta$

$$\boxed{dx'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}}\, dx^{\nu}}$$

obs: $\frac{\partial}{\partial x^{\nu}} = \partial_{\nu}$

$$\boxed{\frac{\partial x'^{\mu}}{\partial x^{\nu}} = \partial_{\nu} x'^{\mu} \equiv S^{\mu}_{\nu}(x)}$$

$$\boxed{dx'^{\mu} = S^{\mu}_{\nu}(x)\, dx^{\nu}}$$ relação Linear

$$\boxed{dx^{\mu} = \frac{\partial x^{\mu}}{\partial x'^{\nu}}\, dx'^{\nu} \equiv (S^{-1})^{\mu}_{\nu}\, dx'^{\nu}} \quad (1)$$

$S^{\mu}_{\nu}(x)$ é uma generalização de $dx'^{\mu} = R^{\mu}_{\nu}\, dx^{\nu}$

mas dependente de **x**!

Obs: $$\boxed{(S^{-1})^{\mu}_{\rho}\, S^{\rho}_{\nu} = \frac{\partial x^{\mu}}{\partial x'^{\rho}} \frac{\partial x'^{\rho}}{\partial x^{\nu}} = \delta^{\mu}_{\nu}}$$

Ex: $dx'^1 = dr = \dfrac{x\,dx + y\,dy}{\sqrt{x^2+y^2}}$

$dx'^2 = d\theta = \dfrac{x\,dx - y\,dy}{x^2+y^2}$

$S^1_1 = \dfrac{x}{\sqrt{x^2+y^2}}$ $\quad S^1_2 = \dfrac{y}{\sqrt{x^2+y^2}}$

$S^2_1 = \dfrac{-y}{x^2+y^2}$ $\quad S^2_2 = \dfrac{x}{x^2+y^2}$

E a métrica?

$ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu = g'_{\rho\sigma}(x')\, dx'^\rho dx'^\sigma$

$= g_{\mu\nu}(x) \dfrac{\partial x^\mu}{\partial x'^\rho} \dfrac{\partial x^\nu}{\partial x'^\sigma} dx'^\rho dx'^\sigma$

$\to$ Mesmo ponto p descrito em 2 coordenadas diferentes: x^μ e x'^μ

$$\boxed{g'_{\rho\sigma}(x') = g_{\mu\nu}(x) (S^{-1})^\mu_\rho (S^{-1})^\nu_\sigma} \quad (2)$$

De (1) e (2) temos que

$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$ fica invariante

$$\partial_\mu = \frac{\partial}{\partial x^\mu}$$

$$\partial'_\mu = \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu} = (S^{-1})^\nu{}_\mu \, \partial_\nu$$

Matrizes

$$g'_{\rho\sigma}(x') = g_{\mu\nu}(x)\,(S^{-1})^\mu{}_\rho\,(S^{-1})^\nu{}_\sigma$$

$$= (S^{-1})^\mu{}_\rho\, g_{\mu\nu}(x)\,(S^{-1})^\nu{}_\sigma$$

$$= \left((S^{-1})^T\right)_\rho{}^\mu\, g_{\mu\nu}(x)\,(S^{-1})^\nu{}_\sigma$$

$$\boxed{g'(x') = (S^{-1})^T\, g(x)\, S^{-1}}$$

Lembrando:

$$R^T R = \mathbb{1}$$
$$\hookrightarrow \mathbb{1} = R^T\,\mathbb{1}\,R$$

uma rotação é uma T.G.C. que deixa a métrica Euclideana ($\mathbb{1}$) invariante!

Métrica inversa:

$$g^{\mu\nu} g_{\nu\rho} = \delta^\mu{}_\rho$$

ou $(g^{-1})^{\mu\nu}$

$$g'^{\mu\nu}(x') = S^\mu{}_\rho\, S^\nu{}_\sigma\, g^{\rho\sigma}(x)$$

(HW)

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix} \qquad g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1/r^2\sin^2\theta \end{pmatrix}$$

Escalares, vetores, tensores sob T.G.C.:

vetor "contra-variante"

$$W'^{\mu}(x') = S^{\mu}_{\ \nu}(x)\, W^{\nu}(x)$$

vetor "covariante"

$$W'_{\mu}(x') = W_{\rho}(x)\, (S^{-1})^{\rho}_{\ \mu}(x)$$

escalar

$$\underbrace{\phi'(x') = \phi(x)}$$

mesmo ponto P
nas descrições x^{μ}, x'^{μ}

$$V'_{\mu}(x')\, W^{\mu\prime}(x') = V_{\rho}(x) \underbrace{(S^{-1})^{\rho}_{\ \mu}\, S^{\mu}_{\ \nu}}_{\delta^{\rho}_{\nu}} W^{\nu}(x) = V_{\rho}(x)\, W^{\rho}(x)$$

Em geral $S^{-1} \neq S^{T}$

Abaixar / subir índices:

$$W^{\nu} \to W_{\mu} = g_{\mu\nu} W^{\nu}$$

$$V_{\mu} \to V^{\mu} = g^{\mu\nu} V_{\nu}$$

$$V^{\nu} = g^{\nu\mu} V_{\mu}$$

Verificar

$$W'_{\rho} = g'_{\rho\sigma} W'^{\sigma} = g_{\mu\nu} (S^{-1})^{\mu}_{\ \rho} \underbrace{(S^{-1})^{\nu}_{\ \sigma}\, S^{\sigma}_{\ \lambda}}_{\delta^{\nu}_{\ \lambda}} W^{\lambda}$$

$$= g_{\mu\nu} (S^{-1})^{\mu}_{\ \rho} W^{\nu} = (S^{-1})^{\mu}_{\ \rho} W_{\mu} \quad \blacksquare$$

Ex - Resolver

$$g_{\mu\rho} A^{\rho} = B_{\mu}$$

$$A^{\rho} = ?$$

$\sigma = 1 \dots D$

$$\sum_{\mu} g^{\sigma\mu} \left(g_{\mu\rho} A^{\rho} = B_{\mu} \right) \Rightarrow \underbrace{\delta^{\sigma}_{\rho} A^{\rho}}_{A^{\sigma}} = g^{\sigma\mu} B_{\mu}$$

Area e volume

$d^D x$ não se transforma apropriadamente sob uma T.G.C. : $x^{\mu} \longrightarrow x'^{\mu}$

e.g. $dx\,dy\,dz \neq dr\,d\theta\,d\varphi$

→ Precisamos de um det. Jacobiano

$$\boxed{d^D x = d^D x' J} \qquad J = \det\left(\frac{\partial x^{\mu}}{\partial x'^{\varphi}}\right)$$

$$\det\left(g'_{\rho\sigma}(x') = g_{\mu\nu}(x) \frac{\partial x^{\mu}}{\partial x'^{\rho}} \frac{\partial x^{\nu}}{\partial x'^{\sigma}} \right)$$

$$\det g'_{\rho\sigma}(x') \equiv g' = \underbrace{\det(g_{\mu\nu}(x))}_{\equiv g} \cdot J \cdot J$$

$$\boxed{g' = g J^2} \; (*) \implies$$

$$\implies d^D x \sqrt{g} = \underbrace{d^D x' J}_{d^D x}\ (**) \;\; \underbrace{\frac{\sqrt{g'}}{J}}_{\sqrt{g}}\ (*) = d^D x' \sqrt{g'}$$

Então

$$\underbrace{dx\,dy\,dz}\neq\underbrace{dr\,d\theta\,d\varphi}$$

$$g_{\mu\nu}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\qquad g_{\mu\nu}=\begin{pmatrix}1&0&0\\0&r^2&0\\0&0&r^2\sin^2\theta\end{pmatrix}$$

$\Rightarrow g=1$ $\qquad\qquad \Rightarrow g'=r^4\sin^2\theta$

$$\therefore\ dx\,dy\,dz\sqrt{1}=dr\,d\theta\,d\varphi\sqrt{r^4\sin^2\theta}$$
$$=r^2\sin\theta\,dr\,d\theta\,d\varphi$$

<u>Divergência, Laplaciano</u>

$$\underbrace{\nabla\cdot\vec{V}\qquad\nabla^2\phi}$$

Dependem das coordenadas usadas

1) $\partial_\mu W^\mu$ não é uma divergência pois não é um escalar

$$\partial_\mu W^\mu\to\partial'_\mu W'^\mu=\underbrace{(S^{-1})^\nu{}_\mu\,\partial_\nu}_{\partial'_\nu}\underbrace{(S^\mu{}_\rho W^\rho)}_{=W'^\mu}$$

$$=(S^{-1})^\nu{}_\mu(\partial_\nu S^\mu{}_\rho)W^\rho+\underbrace{(S^{-1})^\nu{}_\mu S^\mu{}_\rho}_{=\delta^\nu_\rho}\partial_\nu W^\rho=$$

$$=\partial_\mu W^\mu+\overbrace{(S^{-1})^\nu{}_\mu\underbrace{(\partial_\nu S^\mu{}_\rho)}_{\neq 0\text{ para T.G.C.}}W^\rho}^{\text{extra}}$$

$\Rightarrow$ preciso modificar a definição de divergência

$$\underset{\downarrow\ \text{escalar}}{I}=\int\underbrace{d^Dx\sqrt{g}}_{\text{escalar}}\ \underbrace{W^\mu(x)\,\partial_\mu\phi(x)}_{\text{escalar}}\qquad\text{(HW)}$$

por partes:

$$I = \underbrace{-\int d^Dx\, \partial_\mu\left(\sqrt{g}\, W^\mu(x)\right)\, \phi(x)} + \int \underbrace{d^Dx\, \partial_\mu\left(\sqrt{g}\, W^\mu(x)\, \phi(x)\right)} \longrightarrow 0$$

$$\Downarrow$$

$$\int_{\text{borda}: |x|\to\infty} d^{D-1}x\, \sqrt{g}\, W^\mu(x)\, \phi(x) \qquad \begin{pmatrix} W^\mu \to 0 \\ \phi \to 0 \end{pmatrix}$$

$$-\int \underbrace{d^Dx\, \sqrt{g}}_{\text{escalar}}\, \underbrace{\frac{1}{\sqrt{g}}\, \partial_\mu\left(\sqrt{g}\, W^\mu\right)}_{\text{tbm deve ser um escalar}}\, \overbrace{\phi(x)}^{\text{escalar}}$$

$$\boxed{\frac{1}{\sqrt{g}}\, \partial_\mu\left(\sqrt{g}\, W^\mu\right) \equiv D_\mu W^\mu}$$

Exemplo: Coord. esf.

$$\sqrt{g} = r^2 \sin\theta$$

$$D_\mu W^\mu = \partial_\mu W^\mu + \frac{1}{\sqrt{g}}\left(\partial_\mu \sqrt{g}\right) W^\mu$$

$$= \partial_r W^r + \partial_\theta W^\theta + \partial_\varphi W^\varphi$$

$$+ \frac{2}{r} W^r + \frac{\cos\theta}{\sin\theta} W^\theta$$

2) Laplaciano

Truque

$$\underbrace{\int d^D x \sqrt{g}}_{\text{escalar}} \overbrace{g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi}^{\text{escalar (HW)}}$$

$$\underset{\substack{\uparrow \\ \text{por} \\ \text{partes}}}{=} - \int d^D x \frac{\sqrt{g}}{\sqrt{g}} \partial_\mu \left(\sqrt{g} g^{\mu\nu} \partial_\nu \Phi \right) + \text{termos de superfície} \longrightarrow 0$$

$$D^2 \Phi \equiv \frac{1}{\sqrt{g}} \partial_\mu \left(\sqrt{g} g^{\mu\nu} \partial_\nu \Phi \right)$$

Ex: Coord. esf.

$$D^2 \Phi = \left(\partial_r^2 + \frac{2}{r} \partial_r + \frac{1}{r^2} \partial_\theta^2 + \frac{\cos\theta}{r^2 \sin\theta} \partial_\theta + \frac{1}{r^2 \sin^2\theta} \partial_\varphi^2 \right)$$