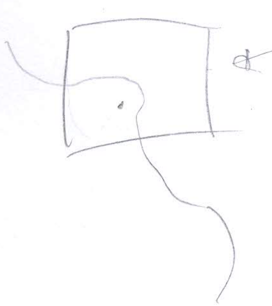
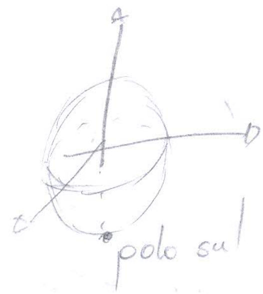
2/3/2016

Espaços curvos



Esfera:



$$x^2+y^2+z^2=L^2 \longrightarrow z=-\sqrt{L^2-x^2-y^2}$$

S: $x=y=0$ $\quad z=-L$

polo sul

$z \to z+L$: polo sul no plano $z=0$

$$z=-\sqrt{L^2-x^2-y^2}+L \qquad (x,y,z)=(0,0,0)$$

Localmente, perto de S: $(x,y)\simeq(0,0)$

concavidade parabólica

$$z \simeq -L\left(1-\frac{1}{2}\frac{x^2}{L^2}-\frac{1}{2}\frac{y^2}{L^2}\right)+L=\frac{1}{2L}(x^2+y^2)$$

Em geral, será:

$$z \simeq \frac{1}{2}ax^2+\frac{1}{2}by^2+cxy$$

$$[a]=[b]=[c]=\frac{1}{\text{comprimento}}$$

$$ds^2 = dx^2 + dy^2 + dz^2$$

$$= dx^2 + dy^2 + \left[(ax+cy)\,dx + (by+cx)\,dy\right]^2$$

$$= \underbrace{g_{xx}}_{g_{xx} = 1+(ax+cy)^2} dx^2 + \underbrace{g_{yy}}_{g_{yy} = 1+(by+cx)^2} dy^2 + 2\underbrace{g_{xy}}_{g_{xy} = (ax+cy)(by+cx)} dx\,dy$$

$$Z \equiv \frac{1}{2}\vec{x}^T M \vec{x} \qquad \vec{x} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad M = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$$

$$\begin{pmatrix} g_{xx} & g_{xy} \\ g_{xy} & g_{yy} \end{pmatrix} \xrightarrow[(x,y)\to 0]{} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

→ Entender onde informações invariantes de curvatura estão escondidas

ex.: curvatura deve ser invariante sob rotações

$$\vec{x} \longrightarrow \vec{x}' = R\vec{x}$$

$$Z \longrightarrow \frac{1}{2}\vec{x}'\underbrace{R^T M R}_{M'}\vec{x}$$

se a curvatura está escondida em M como suspeitamos, ela deve ser invariante sob rotações.

→ deve ser a mesma para M e M'!

→ $\boxed{\text{Traço}}$ e $\boxed{\text{Determinante}}$

Diagonalizar M

$$M = \begin{pmatrix} a & c \\ c & b \end{pmatrix} \longrightarrow \begin{pmatrix} \mu & 0 \\ 0 & \nu \end{pmatrix}$$

$$z = \frac{1}{2}\vec{x}'^T \begin{pmatrix} \mu & 0 \\ 0 & \nu \end{pmatrix} \vec{x}' = \frac{1}{2}(u\ v)\begin{pmatrix} \mu & 0 \\ 0 & \nu \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

$$= \frac{1}{2}\mu u^2 + \frac{1}{2}\nu v^2$$

$\odot$ $\det M = \mu\nu = ab - c^2$

$\odot$ $\frac{(\mathrm{Tr}\, M)^2}{4} = \frac{1}{4}(\mu+\nu)^2 = \frac{1}{4}(a+b)^2$

independentes da escolha de coord.

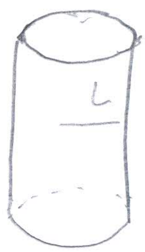
$\begin{cases} \det M \rightarrow \text{curvatura intrínseca} \\ \frac{1}{4}(\mathrm{Tr} M)^2 \rightarrow \quad " \quad \text{extrínseca} \end{cases}$

ex: esfera

$a = b = \frac{1}{L} \qquad c = 0$

$\begin{cases} \det M = a^2 = \frac{1}{L^2} \\ \frac{1}{4}(\mathrm{Tr} M)^2 = a^2 \end{cases}$

ex. um cilíndro tem curvatura extrínseca, mas não tem curv. intrínseca



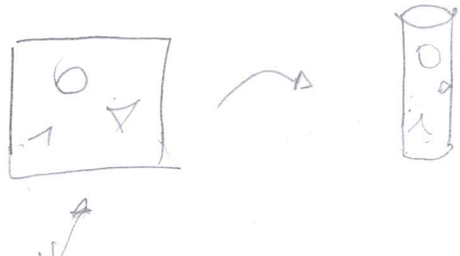
$z = -\sqrt{L^2 - x^2} + L$

indep. de $y \rightarrow b = c = 0, \ a \neq 0$

$\det M = 0$

$\frac{1}{4}(\mathrm{Tr} M)^2 = \frac{1}{4}a^2 \neq 0$

Cilindro:



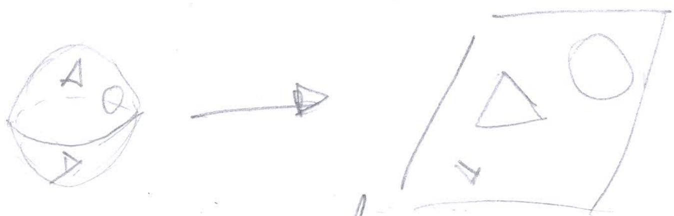
folha de papel

Raios dos círculos e os ângulos do triângulo não mudam

$\Downarrow$

curv. intrínseca $= 0$

esfera



muda ângulos e raios

Esfera: curvatura constante e positiva

$$a^2 > 0$$

curvatura negativa e constante

$$\mu = -\nu$$

sela

$$z = \frac{1}{2}\mu u^2 - \frac{1}{2}\mu \nu^2$$

$$\rightarrow \det M = -u^2 < 0$$

"Embedding" $\rightarrow$ Mergulho

$$\boxed{\underbrace{(x^1, \ldots, x^D)}_{D} \longrightarrow X^A(x^1 \ldots x^D) \quad A = 1 \ldots N}$$

ex: $S^2 \to \mathbb{R}^3$ $\quad (x^1, x^2) = (\theta, \varphi)$

$D=2 \quad N=3$

$$(\underline{X}^1, \underline{X}^2, \underline{X}^3) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

$$(\underline{X}^1)^2 + (\underline{X}^2)^2 + (\underline{X}^3)^2 = 1$$

$$ds^2 = \sum_A (d\underline{X}^A)^2 = \sum_A \frac{\partial \underline{X}^A}{\partial x^\mu} dx^\mu \frac{\partial \underline{X}^A}{\partial x^\nu} dx^\nu$$

$$= \sum_{\mu,\nu} \underbrace{\left(\sum_A \frac{\partial \underline{X}^A}{\partial x^\mu} \frac{\partial \underline{X}^B}{\partial x^\nu} \right)}_{g_{\mu\nu}(\underline{X})} dx^\mu dx^\nu$$

Métrica induzida

Em geral

$$ds^2 = \sum_{A,B} G_{AB}\, d\underline{X}^A d\underline{X}^B$$

Métrica do espaço ambiente

Esfera: $S^2 \to \mathbb{R}^3$ ← ambiente

$$ds^2 = (d\underline{X}^1)^2 + (d\underline{X}^2)^2 + (d\underline{X}^3)^2 = (d\underline{X}^1)^2 + (d\underline{X}^2)^2 + \frac{(\underline{X}^1 d\underline{X}^1 + \underline{X}^2 d\underline{X}^2)^2}{1 - (\underline{X}^1)^2 - (\underline{X}^2)^2}$$

$\mathbb{R}^3$

→ ache a métrica induzida (HW)

Mais em geral → "localmente plano"

(dimensões D)

$$\boxed{g'_{\lambda\sigma}(x') = g_{\mu\nu}(x) \frac{\partial x^\mu}{\partial x'^\lambda} \frac{\partial x^\nu}{\partial x'^\sigma}}$$

$$g_{\mu\nu}(x) \simeq g_{\mu\nu}(0) + A_{\mu\nu\lambda} x^\lambda + B_{\mu\nu\rho\sigma} x^\rho x^\sigma + \dots$$

expandir perto de $x=0$

$$x^\mu = K^\mu{}_\nu x'^\nu + L^\mu{}_{\nu\lambda} x'^\nu x'^\lambda + \dots$$

Quero achar uma transf. que deixe $\begin{cases} g_{\mu\nu}(0) = \delta_{\mu\nu} \\ A_{\mu\nu\lambda} = 0 \end{cases}$

Não vai ser possível, em geral, colocar $B_{\mu\nu\rho\sigma} = 0$

$$\frac{\partial x^\mu}{\partial x'^\nu} = K^\mu{}_\nu + \frac{\partial}{\partial x'^\nu}\left(L^\mu{}_{\rho\lambda} x'^\rho x'^\lambda\right) + \frac{\partial}{\partial x'^\nu}\left(M^\mu{}_{\rho\lambda\sigma} x'^\rho x'^\lambda x'^\sigma\right) + \dots$$

$$= K^\mu{}_\nu + 2 L^\mu{}_{\nu\lambda} x'^\lambda + 3 M^\mu{}_{\nu\lambda\sigma} x'^\lambda x'^\sigma + \dots$$

Simetrias

(HW)

$$\circledast \quad g'_{\lambda\sigma}(x') = \left(g_{\mu\nu}(0) + A_{\mu\nu\lambda} x^\lambda + B_{\mu\nu\lambda\sigma} x^\lambda x^\sigma + \dots\right) \cdot \left(K^\mu{}_\lambda + 2 L^\mu{}_{\lambda\rho} x'^\rho + \dots\right) \cdot \left(K^\nu{}_\sigma + 2 L^\nu{}_{\sigma\zeta} x'^\zeta + \dots\right)$$

$\boxed{x \to x'}$

$$= \left(g_{\mu\nu}(0) + A_{\mu\nu\lambda} K^\lambda{}_\rho x'^\rho + B_{\mu\nu\lambda\sigma} K^\lambda{}_\rho K^\sigma{}_\zeta x'^\rho x'^\zeta + \dots\right) \cdot \left(K^\mu{}_\lambda + \dots\right) \cdot \left(K^\nu{}_\sigma + \dots\right)$$

$\boxed{x' = 0}$

$\boxed{K^\mu{}_\lambda \to D^2 \text{ componentes}}$

$$g'_{\lambda\sigma}(0) = g_{\mu\nu}(0)\, K^\mu{}_\lambda K^\nu{}_\sigma$$

como matrizes: $g' = K^T g K$

preciso usar $\frac{1}{2} D(D+1)$ componentes de $K^\mu{}_\lambda$ para colocar g de forma diagonal

$$\frac{D(D+1)}{2} \qquad g \to \begin{pmatrix} g_{11} & \cdots & 0 \\ & g_{22} & \\ 0 & & g_{DD} \end{pmatrix}$$

"rescaling" / reparametrização

ex: $x^1 \to \frac{x^1}{g_{11}}$

$$\to \begin{pmatrix} g_{11} & & \\ & \ddots & \\ & & g_{DD} \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}$$

Têm $D^2 - \frac{1}{2} D(D+1) = \frac{1}{2} D(D-1)$

componentes de K que não usei

elementos que deixam $\delta_{\mu\nu}$ invariante

rotações

$$g_{\mu\nu}(x) = \delta_{\mu\nu} + \tilde{A}_{\mu\nu\lambda} x^\lambda + \dots$$

novo $A_{\mu\nu\lambda}$ nas novas coord.

$K^\mu{}_\lambda$, $L^\mu{}_{\lambda\sigma} x^\lambda x^\sigma$ com $K^\mu{}_\lambda$ fixado

já usei

$\to$ sim em λ, σ

de componentes de $\tilde{A}_{\mu\nu\lambda}$

$$D \cdot \left(\frac{D(D+1)}{2}\right) \qquad = \qquad \to$$

$\mu\nu \to$ simétrico

$\to$ = # de componentes de $L^{\mu}{}_{\lambda\sigma}$ $\Rightarrow$ posso fixar $\tilde{A}_{\mu\nu\lambda} = 0$

$$\Rightarrow g_{\mu\nu}(x) = \delta_{\mu\nu} + B_{\mu\nu\lambda\sigma} x^{\lambda} x^{\sigma} + \dots$$

$$x^{\mu} = x'^{\mu} + M^{\mu}{}_{\nu\lambda\sigma} x'^{\nu} x'^{\lambda} x'^{\sigma} + \dots$$

$B_{\mu\nu\lambda\sigma}$: $\left(\frac{1}{2} D(D+1)\right)^2$

simétrico. $(\mu\nu)$ $(\lambda\sigma)$

$M^{\mu}{}_{\nu\lambda\sigma}$ (HW)

$D\underbrace{\left(\frac{1}{6} D(D+1)(D+2)\right)}_{(\nu\lambda\sigma)}$

μ

simétrico

$$\underbrace{\left(\frac{1}{2} D(D+1)\right)^2}_{\text{\# de B's}} - \underbrace{D \frac{1}{6} D(D+1)(D+2)}_{\text{\# de M's}}$$

$= \boxed{\frac{D^2}{12}(D+1)(D-1)}$

de componentes de B que não podem ser removidas usando M

$\to$ Curvatura

$D=1 \to 0$

$D=2 \to 1$

$D=3 \to 6$

$D=4 \to 20$

Esquematicamente:

$$\boxed{\text{"}B \sim \partial^2 g + (\partial g)^2\text{"}}$$

Curvatura tem a ver com derivadas segundas da métrica