$h^\mu h^\nu g_{\mu\nu} = 1 = k^\mu k^\nu g_{\mu\nu} \to$ tipo espaço ⊕ ortogonais a ℓ^μ

$h^\mu \ell^\nu g_{\mu\nu} = 0 = k^\mu \ell^\nu g_{\mu\nu}$

11/4/2016

Transformações de Lorentz (T.L.)

- lineares
- coef. constantes

$$\begin{cases} p^\mu \to p'^\mu = \Lambda^\mu{}_\nu p^\nu \\ q^\mu \to q'^\mu = \Lambda^\mu{}_\nu q^\nu \end{cases}$$

$p \cdot q$ devem ser invariantes

$p \cdot q \to p' \cdot q' = \eta_{\mu\nu} p'^\mu q'^\nu$

$= \eta_{\mu\nu} \Lambda^\mu{}_\sigma p^\sigma \Lambda^\nu{}_\rho q^\rho \doteq p \cdot q = \eta_{\sigma\rho} p^\sigma q^\rho$

$$\boxed{\eta_{\mu\nu} \Lambda^\mu{}_\sigma \Lambda^\nu{}_\rho = \eta_{\sigma\rho}}$$

$\Lambda^\mu{}_\sigma \eta_{\mu\nu} \Lambda^\nu{}_\rho = \eta_{\sigma\rho}$

$(\Lambda_\sigma{}^\mu)^T \eta_{\mu\nu} \Lambda^\nu{}_\rho = \eta_{\sigma\rho}$

def de T.L.

$$\boxed{\Lambda^T \eta \Lambda = \eta}$$

forma matricial

mais em geral

$g'(x') = (S^{-1})^T g(x) S^{-1}$

$\hookrightarrow$ T.G.C.

R, Λ são a S^{-1} que preservam $\mathbb{1}, \eta$

Algebra $\to$ Transf. infinitesimais e geradores

$\Lambda^\mu{}_\sigma \simeq \delta^\mu{}_\sigma + \varphi K^\mu{}_\sigma + \mathcal{O}(\varphi^2)$

parâmetro

gerador

Lembre: $R \simeq \mathbb{1} + \theta . J + \mathcal{O}(\theta^2)$

$$\eta_{\mu\nu} \Lambda^{\mu}{}_{\sigma} \Lambda^{\nu}{}_{\rho} = \eta_{\sigma\rho}$$

$$\underbrace{\eta_{\mu\nu} (\delta^{\mu}{}_{\sigma} + \varphi K^{\mu}{}_{\sigma})}(\delta^{\nu}{}_{\rho} + \varphi K^{\nu}{}_{\rho}) = \eta_{\sigma\rho}$$

$$K^{\mu}{}_{\sigma} \, \eta_{\mu\rho} + \eta_{\sigma\nu} K^{\nu}{}_{\rho} = 0 \longrightarrow \boxed{K^T \eta = -\eta K}$$

$$K_{\mu\nu} = \begin{pmatrix} K_{00} & K_{01} & K_{02} & K_{03} \\ K_{10} & K_{11} & K_{12} & K_{13} \\ K_{20} & K_{21} & K_{22} & K_{23} \\ K_{30} & K_{31} & K_{32} & K_{33} \end{pmatrix} \qquad \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\mu, \nu = \underbrace{0}_{\text{Tempo}}, \underbrace{1,2,3}_{\text{espaço}}$

2 direções espaciais

$$\begin{pmatrix} K_{22} & K_{32} \\ K_{23} & K_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} K_{22} & K_{23} \\ K_{32} & K_{33} \end{pmatrix}$$

$$K_{22} = -K_{22} \qquad K_{33} = -K_{33} \qquad K_{32} = -K_{23}$$

$$\rightarrow \quad K_{22} = 0, \quad K_{33} = 0 \Rightarrow J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Rotações!

1 tempo + 1 espaço

$$\begin{pmatrix} K_{00} & K_{10} \\ K_{01} & K_{11} \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} K_{00} & K_{01} \\ K_{10} & K_{11} \end{pmatrix}$$

boost

$$\begin{pmatrix} -K_{00} & K_{10} \\ -K_{01} & K_{11} \end{pmatrix} = -\begin{pmatrix} -K_{00} & -K_{01} \\ K_{10} & K_{11} \end{pmatrix} \Rightarrow \boxed{K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$$

Simétrica

Transf. finitas

$$\varphi = \frac{\phi}{N}$$

infinitesimal (φ), finito (ϕ), grande (N)

$$\begin{pmatrix} K_{00} & K_{01} \\ K_{10} & K_{11} \end{pmatrix}$$

$$\Lambda(\varphi) \simeq 1 + \varphi K = 1 + \frac{\Phi}{N} K$$

$$\Lambda(\Phi) \underset{N\to\infty}{=} [\Lambda(\varphi)]^N \Rightarrow e^{\Phi K} = \sum_{n=0}^{\infty} \frac{\Phi^n K^n}{n!}$$

$$= \left(\sum_{k=0}^{\infty} \frac{\Phi^{2k}}{(2k)!} \right) + K \left(\sum_{k=0}^{\infty} \frac{\Phi^{2k+1}}{(2k+1)!} \right)$$

$$= \cosh \Phi + K \sinh \Phi = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix}$$

Lorentz é um grupo

1) Composição

$$\begin{cases} x^\mu \to \Lambda^\mu{}_\nu x^\nu = x'^\mu \\ x'^\mu \to x''^\mu = \bar\Lambda^\mu{}_\nu x'^\nu \end{cases}$$

$$x''^\mu = \underbrace{\bar\Lambda^\mu{}_\nu \Lambda^\nu{}_\rho}_{\bar{\bar\Lambda}^\mu{}_\rho} x^\rho$$

2) $(\det \Lambda)^2 = 1$

$$\to \det \Lambda = \pm 1 \neq 0$$

$\exists$ inversa

$$(\Lambda^{-1})^\rho{}_\nu = \Lambda_\nu{}^\rho$$

3) $\exists$ indentidade $\Lambda = 1$

Escolhemos

•) $\boxed{\det \Lambda = +1}$

T. L. própria

•) $\eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\mu{}_\sigma = \eta_{\rho\sigma}$

$$(\Lambda^0{}_0)^2 = 1 + \Lambda^i{}_0 \Lambda^i{}_0 \rightarrow \Lambda^0{}_0 = \pm 1$$

$\boxed{\Lambda^0{}_0 = +1}$ T. L. ortocrôna

SO(D): deixa invariante

$$d\vec{x}^2 = \sum_{i=1}^{D} (dx_i)^2$$

SO(m, n) $\quad ds^2 = -\sum_{i=1}^{m} (dx_i)^2 + \sum_{i=m+1}^{n} (dx_i)^2$

Lorentz: SO(1,3) (ou SO(3,1) convenção)

Algebra: $[J_{\mu\nu}, J_{\rho\sigma}] = i\,[\eta_{\mu\rho} J_{\nu\sigma} + \eta_{\nu\sigma} J_{\mu\rho} - \eta_{\nu\rho} J_{\mu\sigma} - \eta_{\mu\sigma} J_{\nu\rho}]$

("a algebra fecha") (HW)

$D=3 \quad SO(1,3) \quad i=(1,2,3)$ ou (x,y,z)

1t+1e $\quad J_{0i} = K_i \quad$ 3 boosts

2e = $\quad J_{jk} \Rightarrow \frac{1}{2}\epsilon_{ijk} J_{ik} = J_i \quad$ 3 rotações

$\boxed{\text{6 geradores}}$

$\boxed{\begin{array}{l}\text{3 boost +} \\ \text{3 rotações +} \\ \text{4 translações} \\ \text{de Grupo Poincaré}\end{array}}$

Por exemplo:

$$[J_x, J_y] = [J_{23}, J_{31}] = -i\eta_{33} J_{21} = i J_{12} = i J_z$$

$$[J_z, K_x] = [J_{12}, J_{01}] = -i\eta_{11} J_{20} = i K_y$$

$$[K_x, K_y] = [J_{01}, J_{02}] = i\eta_{00} J_{12} = -i J_z$$

Nota: $R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \qquad (t = x^0, x^1) \to (x^2, x^1)$

$K = \begin{pmatrix} ch\varphi & sh\varphi \\ sh\varphi & ch\varphi \end{pmatrix}$

Rotação de Wick

$$\begin{cases} t = x^0 = i x^2 \\ \varphi = i\theta \end{cases}$$

$$\begin{cases} ch\varphi = \dfrac{e^{\varphi} + e^{-\varphi}}{2} \to \dfrac{e^{i\theta} + e^{-i\theta}}{2} = \cos\theta \\ sh\varphi = \dfrac{e^{\varphi} - e^{-\varphi}}{2} \to \dfrac{e^{i\theta} - e^{-i\theta}}{2} = i\sin\theta \end{cases}$$

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \underbrace{\begin{pmatrix} ch\varphi & sh\varphi \\ sh\varphi & ch\varphi \end{pmatrix}}_{SO(1,1)} \begin{pmatrix} t \\ x \end{pmatrix}$$

$\downarrow$

$$\begin{pmatrix} ix_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} ix_2 \\ x_3 \end{pmatrix}$$

$$\begin{pmatrix} x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix}$$

$so(2)$

Ação para uma partícula não-relativística

$$S = \int dt\, L = \int dt \left(\frac{m}{2} \left(\frac{d\vec{x}}{dt} \right)^2 - V(\vec{x}) \right)$$

Por enquanto: $V(\vec{x}) = 0$

Generalização relativística?

$$\sqrt{a^2 - \varepsilon^2} \underset{\varepsilon \ll 1}{\cong} a - \frac{\varepsilon^2}{2a} + \dots \Rightarrow \frac{\varepsilon^2}{2a} = a - \sqrt{a^2 - \varepsilon^2}$$

Por motivos dimensionais → precisamos introduzir c

$$\frac{(d\vec{x})^2}{2dt} = \frac{c\,(d\vec{x})^2}{2c\,dt} = c\left(c\,dt - \sqrt{\underbrace{c^2dt^2 - d\vec{x}^2}_{\text{tempo próprio!}}} + \dots\right)$$

$\varepsilon \leftrightarrow d\vec{x}$ Não-relativístico

$a \leftrightarrow c\,dt$ $\varepsilon \ll a$

$$\boxed{S = -mc \int \sqrt{c^2dt^2 - d\vec{x}^2} + \cancel{mc^2 \int dt}^{\,=0}} \qquad \boxed{\delta t_i = \delta t_f = 0}$$

$$\boxed{S_{\text{rel.}} = -mc^2 \int dt \sqrt{1 - \left(\frac{d\vec{x}}{c\,dt}\right)^2}} = -mc^2 \int ds$$

inv. Lorentz ⊕

inv. reparametrização

Teste: limite N.Rel.

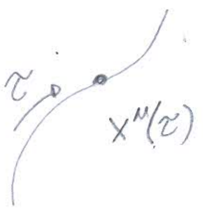
$$S_{rel} \simeq -mc^2 \int dt \left(1 - \frac{1}{2}\left(\frac{d\vec{x}}{c\,dt}\right)^2 + \dots\right)$$

$$= \underbrace{-mc^2 \int dt}_{\text{"alguma" energia}} + \frac{m}{2}\int dt \left(\frac{d\vec{x}}{dt}\right)^2 + \dots \quad \checkmark$$

Que tal $m=0$? $(c=1)$

$$S_{rel} = 0$$

→ Vamos introduzir um objeto auxiliar "sem dinâmica"



$e(\tau)$ métrica ao longo da linha de mundo

$$\tilde{S}_{rel} = \frac{1}{2}\int dt \left(e^{-1}\dot{x}^2 - m^2 e\right)$$

↳ $\dot{x}^2 = \eta_{\mu\nu}\dfrac{dx^\mu}{d\tau}\dfrac{dx^\nu}{d\tau}$

eq. do movimento de $e(\tau)$: (algébrica, não diferenciável)

$$0 = \frac{d}{dt}\left(\frac{\delta L}{\delta \dot{e}}\right) = \frac{\delta L}{\delta e} \to m^2 e^2 + \dot{x}^2 = 0$$

$$e^2 = -\frac{\dot{x}^2}{m^2}$$

$$\tilde{S}_{rel} = \frac{1}{2}\int d\tau \left(e^{-1}\dot{x}^2 - m^2 e\right) = \frac{1}{2}\int d\tau \left(\sqrt{\frac{-m^2}{\dot{x}^2}}\,\dot{x}^2 - m^2\sqrt{\frac{-\dot{x}^2}{m^2}}\right)$$

$$= -m\int d\tau \sqrt{-\dot{x}^2} = -m\int d\tau \sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}} =$$

$$= -m \int \sqrt{\eta_{\mu\nu}\, dx^\mu dx^\nu} = -m \int \sqrt{dt^2 - d\vec{x}^2}$$

S_{rel} e $\tilde{S}_{rel}$ são equivalentes na "$\underbrace{\text{camada de massa}}$" de e.=

= usando as eq. do mov. de. e.

on shell