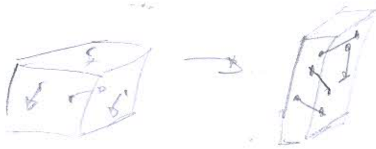
18/4/2016

Densidade de energia

$\rho(t,\vec{x})$ é um 3-escalar na física não rel.
(= sob $SO(3)$)

sob uma T. L.? Energia/volume 

1) Caixa tem contração de Lorentz

$$\to \frac{1}{\sqrt{1-v^2}} \quad (c=1)$$

2) energia das partículas muda

$$\to \left(\frac{1}{\sqrt{1-v^2}}\right)$$

" $\frac{1}{\sqrt{1-v^2}} \leftrightarrow$ componente 0 " $\longrightarrow$ ρ deve ser promovido à componente 00 de um tensor

$$T^{00} \to (T^{00})' = \Lambda^0{}_\mu \Lambda^0{}_\nu T^{\mu\nu} =$$

$$= \Lambda^0{}_0 \Lambda^0{}_0 T^{00} = \left(\frac{1}{\sqrt{1-v^2}}\right)^2 T^{00}$$

$\hookrightarrow T^{00} \neq 0$

$T^{\mu\nu}$ = tensor de energia-momento ou tensor de "stress"

$$T^{\mu\nu} = \sum_a \int d\tau_a \, m_a \frac{dq_a^\mu}{d\tau_a}\frac{dq_a^\nu}{d\tau_a}\, \delta^{(4)}(x - q_a(\tau_a))$$

•) $T^{\mu\nu} = T^{\nu\mu}$

$\frac{4\cdot 5}{2} = 10$ componentes $\left(\frac{D(D+1)}{2}\right)$

D = # dim espaço e tempo

•) $\partial_\mu T^{\mu\nu} = 0$

$T^{00} \rightarrow$ densi de energia

$$\int d\vec{x}^{\,0}\, T^{0\nu} = \sum_a \int d\tau_a \int_V d\vec{x}^{\,0}\, \frac{dq_a^0}{d\tau_a} \underbrace{\frac{dq_a^\nu}{d\tau_a}}_{/\!/\, p_a^\nu}\, \underbrace{\delta^{(4)}(x - q_a(\tau_a))}_{\delta(x^0 - q_a^0(\tau_a))\,\delta^{(3)}(\vec{x} - \vec{q}(\tau_a))}$$

$$= \sum_a \int d\tau_a \frac{dq_a^0}{d\tau_a} p_a^\nu\, \delta(x^0 - q_a^0(\tau_a)) = \sum_a p_a^\nu = P^\nu$$ momento total em V

T^{ij} : $$\frac{d}{dt} P^\nu(t) = \int_V d\vec{x}^{\,0} \frac{\partial}{\partial t} T^{0\nu} = -\int_V d\vec{x}^{\,0} \frac{\partial T^{i\nu}}{\partial x^i} \overset{\text{stokes/Gauss}}{=} -\int_{\partial V} dS_i\, T^{i\nu}$$

∂V "borda do espaço V

$$\begin{cases} \partial_\mu T^{\mu\nu} = 0 \\ \partial_0 T^{0\nu} + \partial_i T^{i\nu} = 0 \end{cases}$$

$\boxed{\nu = j}$ $$\underbrace{\frac{d}{dt} p^j(t)}_{\text{força}} = \underbrace{-\int_{\partial x} dS_i\, T^{ij}}_{T^{ij} \cdot \text{area}}$$

"$T^{ij} = \dfrac{\text{força}}{\text{area}} = \text{pressão}$"

pressão na direção j exercida por uma força na direção i

Ação de uma partícula rel. livre (no espaço-tempo de Minkowski)

$$(c=1)\begin{cases} S_{rel} = -m \int \sqrt{-\eta_{\mu\nu}\, dx^\mu dx^\nu} \\ \quad = -m \int \sqrt{dt^2 - d\vec{x}^{\,2}} \end{cases}$$

Não tem interação!

Como introduzir V de volta?

$\boxed{\text{Propostas}}$.

Duas opções:

(E) $S = -\int \left(m\sqrt{-\eta_{\mu\nu}dx^\mu dx^\nu} + V(\vec{x})dt\right)$

dt e dt^2 não são inv. de Lorentz

(G) $S = -m \int \sqrt{\left(1 + \frac{2V}{m}\right)dt^2 - d\vec{x}^{\,2}}$

Problema: não são inv. de Lorentz!

(Vamos resolver depois...)

Ⓞ limite não rel. funciona!

Ⓔ ok obvio

Ⓖ $|d\vec{x}| \ll dt \quad (|\vec{v}| \ll 1 = c)$

$$S_G = -m \int \sqrt{1 + \frac{2V}{m}}\, dt - \frac{d\vec{x}^2}{2\sqrt{1+\frac{2V}{m}}\, dt}$$

$\boxed{V \ll m}$

$\sqrt{1+\frac{2V}{m}} \simeq 1 + \frac{V}{m}$

$$S_G \simeq \underbrace{-m \int \left(1 + \frac{V}{m}\right) dt}_{} + \underbrace{\int dt\, \frac{m}{2}\left(\frac{d\vec{x}}{dt}\right)^2}_{}$$

$$\cancel{-m\int dt} - \int V(\vec{x})\, dt$$

ok

ok

Precisamos de "promoções"

Ⓔ $V(\vec{x})dt \rightarrow A_0\, dt \rightarrow A_0 d - A_i dx_i$

Eletromag.

$\boxed{\int A_\mu dx^\mu}$ invariante de Lorentz!

$$\boxed{S_E = \int \left(-m\sqrt{-\eta_{\mu\nu}\, dx^\mu dx^\nu} + A_\mu(x)\, dx^\mu \right)}$$

Gravidade

$A_0 = -V$

Ⓖ $-\eta_{\mu\nu}\, dx^\mu dx^\nu \longrightarrow -\left(1 + \frac{2V}{m}\right) dt^2 + d\vec{x}^2$

$$\downarrow$$

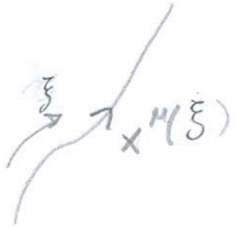
$g_{\mu\nu}\, dx^\mu dx^\nu$

$$S_G = -m \int \sqrt{-g_{\mu\nu}(x)\, dx^\mu dx^\nu}$$

inv. de Lorentz

Eletromagnetismo

$$S_E = -m \underbrace{\int \sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\xi}\frac{dx^\nu}{d\xi}}\,d\xi}_{(i)} + \underbrace{\int A_\mu(x)\,\frac{dx^\mu}{d\xi}\,d\xi}_{(ii)}$$



Eq. de Euler-Laugrange

$$(i)\quad \delta\Big(-m\int d\xi\, \overset{1/m}{\sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\xi}\frac{dx}{d\xi}}}\Big)$$

$$= m\int d\tau\, \eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{d}{d\tau}\delta x^\nu = -m\int d\tau\, \eta_{\mu\rho}\frac{d^2x^\mu}{d\tau^2}\delta x^\rho$$

$\xi = \tau : \boxed{L = M}$

$$(ii)\quad \delta\int d\tau\, A_\mu \frac{dx^\mu}{d\tau} = \int d\tau\Big(A_\mu\frac{d\,\delta x^\mu}{d\tau} + (\partial_\nu A_\mu)\,\delta x^\nu\frac{dx^\mu}{d\tau}\Big) =$$

$$\boxed{\begin{aligned}\delta A_\mu(x) &= \frac{\partial A_\mu}{\partial x^\nu}\,\delta x^\nu\\ &= \partial_\nu A_\mu\,\delta x^\nu\end{aligned}}$$

$$= \int d\tau\Big(-\underbrace{\Big(\frac{d}{d\tau}A_\mu\Big)}_{//\ \partial_\nu A_\mu \frac{dx^\nu}{d\tau}}\delta x^\mu + \partial_\nu A_\mu\frac{\partial x^\mu}{d\tau}\delta x^\nu\Big)$$

$$= \int d\tau\Big(\partial_\mu A_\nu\frac{dx^\nu}{d\tau}\delta x^\mu - \partial_\nu A_\mu\frac{dx^\nu}{d\tau}\delta x^\mu\Big)$$

$$= \int \underbrace{\big(\partial_\mu A_\nu - \partial_\nu A_\mu\big)}_{\equiv F_{\mu\nu}}\frac{dx^\mu}{d\tau}\,\delta x^\mu$$

$$\delta S_E = \int d\tau\Big(-m\,\eta_{\mu\rho}\frac{d^2x^\mu}{d\tau^2}\delta x^\rho + \overbrace{F_{\mu\nu}\frac{dx^\nu}{d\tau}\delta x^\mu}^{F^\mu{}_\nu \frac{dx^\nu}{d\tau}\overbrace{\eta_{\mu\rho}\delta x^\rho}^{\delta x_\mu}}\Big) = \downarrow$$

$$\to \int d\tau \underbrace{\left(-m\frac{d^2x^\mu}{d\tau^2} + F^\mu{}_\nu \frac{dx^\nu}{d\tau}\right)}_{=0} \eta_{\mu\rho}\, \delta x^\rho$$

$$\boxed{m\frac{d^2x^\mu}{d\tau^2} = F^\mu{}_\nu \frac{dx^\nu}{d\tau} \equiv \mathcal{F}^\mu}$$

Força de Lorentz

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{pmatrix} 0 & F_{01} & F_{02} & F_{03} \\ -F_{01} & 0 & F_{12} & F_{13} \\ -F_{02} & -F_{12} & 0 & F_{23} \\ -F_{03} & -F_{13} & -F_{23} & 0 \end{pmatrix}$$

$$\boxed{F_{0i} = -E_i}$$

$$\boxed{\epsilon_{ijk} F_{jk} \equiv B_i}$$

$$\Longrightarrow \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

$$\begin{cases} B^3 = F^{12} = F_{12} = B^z \\ B^2 = F^{31} = F_{31} = B^y \end{cases}$$

$$\begin{cases} \vec{E} = \text{campo elétrico} \\ \vec{B} = \text{" magnético} \end{cases}$$

Notem: $\vec{E}$ e $\vec{B}$ NÃO são as componentes espaciais de um 4-vetor!

São unificados em um tensor de ordem 2 antissimétrico!!

Ex: 2+1 dim (t, x, y)

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y \\ E_x & 0 & B_z \\ E_y & -B_z & 0 \end{pmatrix}$$

$B_z \Rightarrow B$ é escalar!

$\mu = 1$

$$m\frac{d^2x^1}{d\tau^2} = F^1{}_\nu \frac{dx^\nu}{d\tau}$$

$$\left(\frac{d}{d\tau}\left(m\frac{dx^1}{d\tau}\right) = \frac{d}{d\tau}p^1 = -F^{10}\frac{dx^0}{d\tau} + F^{12}\frac{dx^2}{d\tau} + F^{13}\frac{dx^3}{d\tau}\right) \times \frac{d\tau}{dx}$$

$$= E^1\frac{dx^0}{d\tau} + B^3\frac{dx^2}{d\tau} + B^2\frac{dx^3}{d\tau}$$

$n=2$

$n=3$

$\vdots$

HW

$$\boxed{\frac{d\vec{p}}{dt} = \vec{E} + \vec{v} \times \vec{B}}$$

$\hookrightarrow \frac{d\vec{x}}{dt}$

<u>Maxwell</u>

Ação para A_μ? $\quad A_\mu \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

A_μ ainda não é dinâmico

$$S = \int\left(-m\sqrt{-\eta_{\mu\nu}\, dx^\mu dx^\nu} + eA_\mu\, dx^\mu\right) + ?$$

Como deixar A_μ dinâmico?

$\int A_\mu dx^\mu$ é invariante de Lorentz,

Também tem uma outra invariança

$$\boxed{A_\mu \to A_\mu + \partial_\mu \Lambda(x)} \quad \int A_\mu\, dx^\mu \to \int A_\mu dx^\mu + \int (\partial_\mu \Lambda) dx^\mu = \int A_\mu dx^\mu + \int d\Lambda = \int A_\mu dx_\mu + \underbrace{\Lambda\Big|_{\tau=-\infty}^{\tau=+\infty}}_{\to 0}$$

Transf. de <u>Gauge</u> $\quad \Lambda \to 0$

(Calibre) $\quad (\tau \to \pm\infty)$

$F_{\mu\nu}$ é inv. de calibre

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \to \partial_\mu(A_\nu + \partial_\nu \Lambda) - \partial_\nu(A_\mu + \partial_\mu \Lambda)$$

$$= F_{\mu\nu} + \underbrace{\partial_\mu \partial_\nu \Lambda - \partial_\nu \partial_\mu \Lambda}_{=0}$$

$F_{\mu\nu} F^{\mu\nu}$

$$\boxed{S = \int\left(-m\sqrt{-\eta_{\mu\nu}\, dx^\mu dx^\nu} + eA_\mu dx^\mu\right) - \frac{1}{4}\int d^4x\, F_{\mu\nu} F^{\mu\nu}}$$

similar à:

$$S_{newton} = \int dt \left(\frac{1}{2} m \left(\frac{dq}{dt}\right)^2 - \int d\vec{x}\, m\,\delta^{(3)}(x - q(\tau))\,\Phi\right) - \int d\vec{x}\, \frac{1}{8\pi G}(\nabla \Phi)^2$$

Eq. do movimento

Variação com respeito à partícula

OK → Lorentz

Variação com respeito à A_μ

(i) $\delta\left(\int d^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right)\right) = -\frac{1}{4}\int d^4x\, \underbrace{\delta\left(F_{\mu\nu}F^{\mu\nu}\right)}_{2F_{\mu\nu}\delta F^{\mu\nu} = 4F^{\mu\nu}\partial_\mu \delta A_\nu \ (HW)} =$

int. por partes

$= \int d^4x\, \partial_\mu F^{\mu\nu}(x)\, \delta A_\nu(x)$ (HW)

(ii) $A_\mu(\vec{x}) = \int d^4x\, \delta^{(4)}(x-\bar{x})\, A_\mu(x)$

$\bar{x}$

$\int A_\mu(\bar{x}) \frac{d\bar{x}^\mu}{d\tau} d\tau = \int d^4x\, d\tau\, A_\mu(x)\, \delta^{(4)}(x-\bar{x}) \frac{d\bar{x}^\mu}{d\tau}$

$\delta(\quad) = \int d^4x \Big(\underbrace{e\int d\tau\, \delta^{(4)}(x-\bar{x}) \frac{d\bar{x}^\mu}{d\tau}}_{J^\mu(x)}\Big)\, \delta A_\mu(x)$

(ii) $\int d^4x\, J^\mu(x)\, \delta A_\mu(x)$

(i) + (ii) : $\boxed{\partial_\mu F^{\mu\nu} = -J^\nu}$

Eq. de Maxwell

$\boxed{\nu = 0}$

$\partial_i F^{i0} = -\partial_i E^i = -\nabla\cdot\vec{E} = -J^0$

$\boxed{\nabla\cdot\vec{E} = \rho}$

$\boxed{\nu = i}$

$\boxed{\nabla\times\vec{B} = \frac{\partial\vec{E}}{\partial t} + \vec{J}}$ (HW)

As eq. homogeneas são identidades

$\epsilon^{\mu\nu\lambda\sigma} : \epsilon^{0123} = +1$

$\epsilon^{\mu\nu\lambda\sigma}\partial_\nu F_{\lambda\sigma} = \epsilon^{\mu\nu\lambda\sigma}\left(\partial_\nu\partial_\lambda A_\sigma - \partial_\nu\partial_\sigma A_\lambda\right)$

$$= 2\epsilon^{\mu\nu\lambda\sigma}\partial_\nu\partial_\lambda A_\sigma = 0$$

$$\boxed{\mu=0} \rightarrow \nabla\cdot\vec{B} = 0$$

$$\boxed{\mu=i} \rightarrow \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \quad (HW)$$

$$\partial_\nu\left(\partial_\mu F^{\mu\nu} = -J^\nu\right) \rightarrow \partial_\nu\partial_\mu F^{\mu\nu} = 0$$

$$\downarrow$$

$$\boxed{\partial_\nu J^\nu = 0}$$

Conservação de corrente