2/5/2016

Tensores sob T.G.C.

T.G.C. = $dx'^{\mu} = \frac{\partial x'^{\mu}}{dx^{\nu}} dx^{\nu} \equiv S^{\mu}{}_{\nu}(x)\, dx^{\nu}$

Vetores sob T.G.C. $\quad W'^{\mu}(x) = S^{\mu}{}_{\nu}(x)\, W^{\nu}(x)$

mesmo ponto P nas coord. x^{μ} e x'^{μ}

Tensor sob TGC $\quad T'^{\mu\nu}(x') = S^{\mu}{}_{\rho}(x)\, S^{\nu}{}_{\sigma}(x)\, T^{\rho\sigma}(x)$

indices contravariantes (= up)

Ex. de tensor com indices covariantes (= down): métrica

$$g'_{\rho\sigma}(x') = g_{\mu\nu}(x)\, (S^{-1})^{\mu}{}_{\rho}(x)\, (S^{-1})^{\nu}{}_{\sigma}(x)$$

index contrav $\to S$

" covariante $\to S^{-1}$

*) $W_{\mu} = g_{\mu\nu} W^{\nu}$

vetor cov. associado ao vetor contrav. W^{ν}

Teste:

$$w'_{\rho} = g'_{\rho\sigma} W'^{\sigma} = \underbrace{g_{\mu\nu} (S^{-1})^{\mu}{}_{\rho} (S^{-1})^{\nu}{}_{\sigma}}_{g'_{\rho\sigma}} \underbrace{S^{\sigma}{}_{\lambda} W^{\lambda}}_{W'^{\sigma}}$$

$(S^{-1})^{\nu}{}_{\sigma} S^{\sigma}{}_{\lambda} = \delta^{\nu}_{\lambda}$

*) $W_\mu V^\mu$ é escalar

$$\overset{\shortparallel}{g_{\mu\nu}} W^\mu V^\nu = W^\mu V_\mu = \underbrace{g^{\mu\nu} W_\mu V_\nu}_{= V_\mu W^\mu} = W_\mu (S^{-1})^\mu{}_\rho \quad \underline{\text{OK}}$$

$$\text{teste: } W'_\rho V'^\rho = W_\mu \underbrace{(S^{-1})^\mu{}_\rho S^\rho{}_\nu}_{\delta^\mu_\nu} V^\nu = W_\nu V^\nu$$

escalar: $\phi'(x') = \phi(x)$

*) $\partial_\mu \phi(x)$ é um vetor

$$\partial'_\mu \phi'(x') = \frac{\partial \phi'(x')}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \cdot \frac{\partial \phi(x)}{\partial x^\nu} = (S^{-1})^\nu{}_\mu \, \partial_\nu \phi(x)$$

$$\begin{cases} dx^\mu \to \text{vetor contrav.} \\ \partial_\mu = \dfrac{\partial}{\partial x^\mu} \to \text{'' cov.} \end{cases}$$

*) $\partial_\lambda W^\mu$ é um tensor? Não

$$\partial'_\lambda W'^\mu = \underbrace{\frac{\partial x^\nu}{\partial x'^\lambda} \frac{\partial}{\partial x^\nu}}_{\partial'_\lambda} \underbrace{\left(S^\mu{}_\rho W^\rho \right)}_{W'^\mu}$$

S depende do x!

$$= (S^{-1})^\nu{}_\lambda \frac{\partial}{\partial x^\nu} \left(S^\mu{}_\rho(x) \, W^\rho \right)$$

$$= (S^{-1})^\nu{}_\lambda \, S^\mu{}_\rho(x) \, \partial_\nu W^\rho + \underbrace{(S^{-1})^\nu{}_\lambda \left(\frac{\partial}{\partial x^\nu} S^\mu{}_\rho \right) W^\rho}_{\text{extra!}}$$

→ Derivada Covariante

D_μ : $D_\lambda W^\mu$ é um tensor

$$\boxed{D_\lambda W^\mu \to D'_\lambda W'^\mu = (S^{-1})^\nu{}_\lambda\, S^\mu{}_\rho\, D_\nu W^\rho}$$

$D_\mu = ?$ $\quad D_\mu = \partial_\mu + \dots$

para cancelar o pedaço extra

Ansatz

linear em W^μ:

não pode ser um tensor

$$D_\lambda W^\mu = \partial_\lambda W^\mu + \tilde{\Gamma}^\mu{}_{\lambda\nu} W^\nu$$

→ símbolos de Christoffel

$$\boxed{D_\lambda W^\mu = \partial_\lambda W^\mu + \Gamma^\mu_{\lambda\nu} W^\nu}$$

$$\boxed{D_\lambda W_\mu = \partial_\lambda W_\mu - \Gamma^\sigma_{\mu\lambda} W_\sigma}$$

HW: Check!

use que:

Matrizes:

$$M M^{-1} = \mathbb{1}$$

$$\partial(M M^{-1}) = \partial(\mathbb{1}) = 0$$

$$\partial(M) M^{-1} + M \partial M^{-1} = 0$$

$$\boxed{\partial M^{-1} = -M^{-1}(\partial M) M^{-1}}$$

*) $\boxed{D_\lambda g_{\mu\nu} = 0}$ HW

Definição de Γ como "metric connection"

$$\partial_\lambda g_{\mu\nu} - \Gamma^\sigma_{\lambda\mu} g_{\sigma} - \Gamma^\sigma_{\lambda\nu} g_{\sigma}$$

$$D_\lambda T_{\mu\nu}{}^\rho = \partial_\lambda T_{\mu\nu}{}^\rho - \underbrace{\Gamma T}_{\mu} - \underbrace{\Gamma T}_{\nu} + \underbrace{\Gamma T}_{\rho}$$

*) Maxwell no espaço-tempo curvo

Plano : $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

Curvo : $F_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu = \partial_\mu A_\nu - \cancel{\Gamma^\sigma_{\mu\nu} A_\sigma} - (\partial_\nu A_\mu - \cancel{\Gamma^\sigma_{\nu\mu} A_\sigma})$

$= \partial_\mu A_\nu - \partial_\nu A_\mu$ devido à antissimetria de $F_{\mu\nu}$ e à simetria de $\Gamma_{\mu\nu}$

Plano $-\frac{1}{4}\int d^4x \, F_{\mu\nu} F^{\mu\nu} \to F_{\mu\nu} F_{\rho\sigma} \eta^{\mu\rho} \eta^{\nu\sigma}$

Curvo $\boxed{-\frac{1}{4}\int d^4x \sqrt{-\det g}\, g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma}} = S_{maxwell}$

$\boxed{g = \det g_{\mu\nu}}$

*) Diverg. cov.

$$D_\mu W^\mu = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g}\, W^\mu)$$

mas também: (ver aula anterior)

$$D_\mu W^\mu = \partial_\mu W^\mu + \Gamma^\mu_{\mu\nu} W^\nu$$

$\to \boxed{g^{\mu\sigma} \partial_\nu g_{\mu\sigma} = \frac{1}{2} \partial_\nu g}$ HW

Precisar usar

$\log \det M = \operatorname{tr} \log M$

(HW)

Equações de Einstein

$\begin{cases} \to \text{ação de Eistein - Hilbert} \\ \to \text{Variação} \to \text{equações} \end{cases}$

Até agora Newton $\int dt\, d^3\vec{x}\, (\vec{\nabla}\phi)^2 \to \phi$

Maxwell $\int d^4x\, F_{\mu\nu}F^{\mu\nu} \to A_\mu$

1) quadraticas nos campos

2) " nas derivadas

$\Downarrow$

$\begin{cases} \text{ação com 2 derivadas} \\ \text{da métrica, com } (\nabla\phi)^2 \text{ com} \\ \text{limite não - Relativistico} \end{cases}$

$\phi, A_\mu \to 0$ na ausência dos campos

$g_{\mu\nu} \to \eta_{\mu\nu}$

papel análogo a ϕ, A_μ

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$g^{\mu\nu}$ vai aparecer $\to$ $\eta^{\mu\nu} - h^{\mu\nu}$ vai aparecer

$\downarrow$

$(\eta_{\mu\nu} + h_{\mu\nu})^{-1} \to$ série infinita em $h_{\mu\nu}$

$\to$ expectativa: a ação não vai ser quadrática em $h_{\mu\nu}$, mas mais complicada.

*) T.G.C.

a ação deve ser invariante sob T.G.C.

ação $\longleftrightarrow$ curvatura
2 derivadas da métrica
mais complicada que "g^2"
invariante sob T.G.C.

Aliás, não podemos usar $D_\mu g_{\lambda\sigma} = 0$

$\rightarrow$ vai ter combinações de $\partial_\mu g_{\lambda\sigma}$

Curvatura

Exemplos

1D $f(x) \longrightarrow f(x+\delta x) = f(x) + \delta x \dfrac{df(x)}{dx} = \underbrace{\left(1 + \delta x \dfrac{d}{dx}\right)}_{\text{operador de translação}} f(x)$

2D $f(x,y) \longrightarrow f(x+\delta x, y+\delta y) = \underbrace{\left(1 + \delta y \dfrac{d}{dy}\right)}_{\text{transl. em y depois}} \underbrace{\left(1 + \delta x \dfrac{d}{dx}\right)}_{\text{transl. em x primeiro}} f(x,y)$

$$= \left(1 + \delta y \frac{\partial}{\partial y} + \delta x \frac{\partial}{\partial x} + \delta y \, \delta x \frac{\partial}{\partial y}\frac{\partial}{\partial x}\right) f(x,y)$$

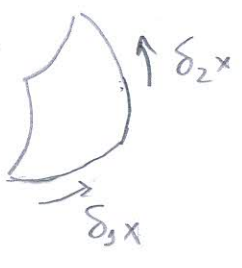
transl. em y primeiro depois em x

$$(x \longleftrightarrow y) \; f(x,y)$$

$$(x \text{ depois } y)\, f(x,y) - (y \text{ depois } x)\, f(x,y) = 0$$

$$\delta x \, \delta y \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right] f(x,y) = 0 \quad \text{óbvio}$$

. Plano $\longrightarrow$ curvo



transladar um vetor $S_\rho(x)$

$$(1+\delta_2 x^\nu D_\nu)(1+\delta_1 x^\mu D_\mu) S_\rho(x) - (1+\delta_1 x^\mu D_\mu)(1+\delta_2 x^\nu D_\nu) S_\rho(x)$$

$$= \delta_2 x^\mu \delta_1 x^\nu \underbrace{[D_\mu, D_\nu] S_\rho(x)}_{\neq 0}$$

$$[D_\mu, D_\nu] S_\rho = (D_\mu D_\nu - D_\nu D_\mu) S_\rho$$

$$D_\mu(D_\nu S_\rho) = \partial_\mu(D_\nu S_\rho) - \Gamma^\sigma_{\mu\nu} D_\sigma S_\rho - \Gamma^\sigma_{\mu\rho} D_\nu S_\sigma$$

$$- D_\nu D_\mu S_\rho = - \partial_\nu(D_\mu S_\rho) + \Gamma^\sigma_{\nu\mu} D_\sigma S_\rho + \Gamma^\sigma_{\nu\rho} D_\mu S_\sigma$$

$$\rightarrow [D_\mu, D_\nu] S_\rho = \partial_\mu(\partial_\nu S_\rho - \Gamma^\sigma_{\nu\rho} S_\sigma) - \Gamma^\sigma_{\mu\rho}(\partial_\nu S_\sigma - \Gamma^\lambda_{\nu\sigma} S_\lambda) - (\mu \leftrightarrow \nu)$$

$$= -(\partial_\mu \Gamma^\sigma_{\nu\rho}) S_\sigma - \Gamma^\sigma_{\nu\rho} \partial_\mu S_\sigma - \Gamma^\sigma_{\mu\rho} \partial_\nu S_\sigma + \Gamma^\sigma_{\mu\rho} \Gamma^\lambda_{\nu\sigma} S_\lambda - (\mu \leftrightarrow \nu)$$

$$= -(\partial_\mu \Gamma^\sigma_{\nu\rho} - \Gamma^\lambda_{\mu\rho} \Gamma^\sigma_{\nu\lambda} - (\mu \leftrightarrow \nu)) S_\sigma = -R^\sigma_{\rho\mu\nu} S_\sigma$$

$$\boxed{R^\sigma_{\rho\mu\nu} = \partial_\mu \Gamma^\sigma_{\nu\rho} + \Gamma^\sigma_{\mu\lambda} \Gamma^\lambda_{\nu\rho} - (\mu \leftrightarrow \nu)} \sim \text{"}\partial g\, \partial g + \partial^2 g\text{"}$$

↳ Tensor de Riemann

1) 2 derivadas da métrica

2) curvatura do espaço-tempo

3) é um tensor

4) é único; imagine que $\exists\ \tilde{R}^\sigma_{\rho\mu\nu}$ que é um tensor com 2 derivadas da métrica

$$\Delta^\sigma_{\rho\mu\nu} \equiv R^\sigma_{\rho\mu\nu} - \tilde{R}^\sigma_{\rho\mu\nu} \xrightarrow[\text{referencial localmente plano}]{} \boxed{\Delta^\sigma_{\rho\mu\nu} = 0}$$

↳ vai ser válido em qlqr referencial.

Propriedades

Ⓐ

1) $\boxed{R^{\lambda}{}_{\rho\mu\nu} = -R^{\lambda}{}_{\rho\lambda\mu}}$ por definição

$\downarrow$

$\boxed{R_{\tau\rho\mu\nu} = g_{\tau\lambda} R^{\lambda}{}_{\rho\mu\nu}} \longrightarrow \underset{\tau}{4} \times \underset{\rho}{4} \times \overset{\mu,\nu}{\left(\frac{4\times 3}{2}\right)} = 96$ componentes Ⓐ

$\left(\frac{4\times 3}{2}\right)$; " Ⓑ $36 \xrightarrow{C} 21 \xrightarrow{D} 20$

(HW)

→ coord. localmente planas

$x \approx 0$

$g_{\tau\mu}(x) = \eta_{\tau\mu} + B_{\tau\mu\lambda\sigma} x^{\lambda} x^{\sigma} + \dots$

$\Longrightarrow \left\{ \begin{array}{l} "\Gamma = 0" \\ \partial\Gamma \neq 0 \end{array} \right. \Longrightarrow R_{\tau\rho\mu\nu} = B_{\tau\nu\mu\rho} - B_{\rho\nu\mu\tau} - (\mu \leftrightarrow \nu) \rightarrow$

$\rightarrow \boxed{R_{\tau\rho\mu\nu} = -R_{\rho\tau\mu\nu}}$ Ⓑ

$\boxed{R_{\tau\rho\mu\nu} = R_{\mu\nu\tau\rho}}$ Ⓒ

$\boxed{R_{\tau\rho\mu\nu} + R_{\tau\mu\nu\rho} + R_{\tau\nu\rho\mu} = 0}$ Ⓓ