4/5/2016

tensor de Riemann
$$\begin{cases} R^{\sigma}{}_{\rho\mu\nu} = \left(\partial_\mu \Gamma^{\sigma}_{\nu\rho} + \Gamma^{\sigma}_{\mu\lambda}\Gamma^{\lambda}_{\nu\rho}\right) - (\mu \longleftrightarrow \nu) \\ R_{\tau\rho\mu\nu} = g_{\tau\lambda} R^{\lambda}{}_{\rho\mu\nu} \end{cases}$$

"$R \sim (\partial g)^2 + \partial^2 g$"

Ação para a métrica

1) escalar

$$\int d^4x \longrightarrow \int d^4x \sqrt{-g} \quad \text{invariante sob T.G.C.}$$

$$g = \det g_{\mu\nu}$$

para $\sqrt{\ } \in \mathbb{R}$

$$\Rightarrow S_{grav} = \int d^4x \sqrt{-g}\, \underbrace{A(x)}_{\text{escalar}} \quad \begin{cases} A(x) = A'(x') \\ A(x) \text{ deve conter 2 derivadas na métrica} \end{cases}$$

$A(x)$ deve ser construido a partir do $R_{\mu\nu\rho\sigma}$

Quantos escalares podemos construir?

$$g^{\mu\nu} R_{\mu\nu\rho\sigma} = 0 = g^{\rho\sigma} R_{\mu\nu\rho\sigma}$$

$$\boxed{g^{\mu\rho} R_{\mu\nu\rho\sigma} \equiv \underbrace{R_{\nu\sigma}}_{= R_{\sigma\nu}}} \rightarrow \boxed{\text{tensor de Ricci}}$$

$$g^{\mu\nu} R_{\mu\nu\rho\sigma} = -g^{\mu\sigma} R_{\mu\nu\sigma\rho} = -R_{\nu\rho}$$

$$\boxed{R \equiv g^{\nu\sigma} R_{\nu\sigma}}$$

↳ escalar de curvatura (de Ricci)

$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$

$\rightarrow dr^2 + r^2 d\theta^2$

$[\theta] = 1$

$$S_{gravid} = \# \int d^4x \sqrt{-g}\, R$$

?

$\boxed{c=1}$

$[S_{grav}] = M.L.$ (e.g. $S = -m \int d\tau$)

$m \to M$, $d\tau \to L$

$\# = \frac{a}{G_N}$ ← num adim

$[g] = 1 \qquad [R] = L^{-2}$

$[\#] = \frac{ML}{L^2} = \frac{M}{L} \qquad [G_N] = \frac{L}{M}$

a é determinado pelo limite N-R $\quad a = \frac{1}{16\pi}$

Ação para métrica

$$\boxed{S_{EH} = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g}\, R}$$

S_{EH} ↓ Einstein - Hilbert

"Navalha de Occam" (+ Simetria)

o equivalente de $\int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}\right)$

A_μ

Equações de Einstein

$$S_{EH} = \frac{1}{16\pi G} \int d^4x \underbrace{\sqrt{-g}}_{I}\, \overbrace{\underbrace{g^{\mu\nu}}_{II} \underbrace{R_{\mu\nu}}_{III}}^{R}$$

$$\delta S_{EH} = \delta S_{EH}\big|_{I} + \delta S_{EH}\big|_{II} + \delta S_{EH}\big|_{III}$$

$$\mathrm{II})\ \frac{1}{16\pi G}\int d^4x\,\sqrt{-g}\,R_{\mu\nu}\,\delta g^{\mu\nu}$$

$$= \boxed{-\frac{1}{16\pi G}\int d^4x\,\sqrt{-g}\,R^{\mu\nu}\,\delta g_{\mu\nu}} \quad \text{"}-g^{\mu\rho}\delta g_{\rho\sigma}\,g^{\sigma\nu}\text{"}$$

$$\delta(MM^{-1}=1) \qquad \delta M^{-1} = -M^{-1}\delta M M$$

$$\mathrm{I})\ \delta(\sqrt{-g}) = \frac{1}{2}\sqrt{-g}\,g^{\mu\nu}\delta g_{\mu\nu}$$

$$\mathrm{HW}\begin{cases}\log\det M = \mathrm{tr}\log M\\ \delta\det M = \det M\,\delta(\mathrm{tr}\log M) = \det M\,\mathrm{tr}(M^{-1}\delta M)\end{cases}$$

$$\delta S_{EH}\Big|_{I} = \frac{1}{16\pi G}\int d^4x\,\sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}R\right)\delta g_{\mu\nu}$$

$$\boxed{\delta S_{EH}\Big|_{I} + \delta S_{EH}\Big|_{II} = \frac{1}{16\pi G}\int d^4x\,\sqrt{-g}\left[R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R\right]\delta g_{\mu\nu}}$$

$$\mathrm{III})\ \delta R_{\mu\nu} = R_{\mu\nu}(g+\delta g) - R_{\mu\nu}(g)$$

$$\Gamma^{\rho}_{\nu\mu} \to \Gamma^{\rho}_{\nu\mu} + \delta\Gamma^{\rho}_{\nu\mu} = \tilde{\Gamma}^{\rho}_{\nu\mu}$$

$\delta\Gamma^{\rho}_{\nu\mu} = \tilde{\Gamma}^{\rho}_{\nu\mu} - \Gamma^{\rho}_{\nu\mu}$ é um tensor (HW)

$$D_\lambda\delta\Gamma^{\rho}_{\nu\mu} = \partial_\lambda\delta\Gamma^{\rho}_{\nu\mu} + \Gamma^{\rho}_{\lambda\sigma}\delta\Gamma^{\sigma}_{\nu\mu} - \Gamma^{\sigma}_{\lambda\nu}\delta\Gamma^{\rho}_{\sigma\mu} - \Gamma^{\sigma}_{\lambda\mu}\delta\Gamma^{\rho}_{\nu\sigma}$$

$$(\mathrm{HW})\ \delta R^{\rho}_{\ \mu\lambda\nu} = D_\lambda(\delta\Gamma^{\rho}_{\nu\mu}) - D_\nu(\delta\Gamma^{\rho}_{\lambda\mu})$$

$$\delta S_{EH}\Big|_{III} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\, g^{\mu\nu}\left(D_\lambda \delta\Gamma^\lambda_{\nu\mu} - D_\nu \delta\Gamma^\lambda_{\lambda\mu}\right)$$

$$= \dots = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\, D_\sigma\left(g_{\mu\nu} D^\sigma \delta g^{\mu\nu} - D_\lambda \delta g^{\sigma\lambda}\right)$$

HW

quando tem borda termos de bordas são importantes

$$= \frac{1}{16\pi G}\int d^4x\sqrt{-g}\, D_\sigma V^\sigma$$

$$= 0$$ Stokes (modulo termos de Borda)

$\boxed{S_{GH} = \text{acão de Gibbons-Hawking}}$

$$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0}$$

Equações de Einstein

(gravidade pura, no vácuo sem matéria)

<u>traço</u>:

$$g^{\mu\nu}\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right) = 0$$

$$R - \frac{1}{2} g^{\mu\nu} g_{\mu\nu} R = 0 \Rightarrow R - \frac{4}{2} R = 0$$

$\boxed{R = 0}$ no vácuo

$\Rightarrow \boxed{R_{\mu\nu} = 0}$ no vácuo

$$R_{\mu\nu} = 0 \;\not\Rightarrow\; R_{\mu\nu\rho\sigma} = 0$$

$R_{\mu\nu\rho\sigma} = 0 \longrightarrow$ espaço - tempo plano

$R_{\mu\nu} = g^{\rho\sigma} R_{\rho\mu\sigma\nu}$ = soma de termos positivos ou negativos

A soma pode ser zero sem os termos da soma serem zero

Estrela esférica + estática

Fora da estrela: vácuo

$$ds^2 = -A(r)\,dt^2 + B(r)\,dr^2 + r^2 d\Omega_2^2$$

$$\begin{cases} A(r) \xrightarrow[r\to\infty]{} 1 - \dfrac{2GM}{r} \\ B(r) \xrightarrow[r\to\infty]{} 1 \end{cases}$$

vácuo $\to$ $\boxed{R_{\mu\nu} = 0}$

não vácuo: $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \neq 0$

$$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0}$$

$$R_{\nu\rho} = \left(\partial_\sigma \Gamma^\sigma_{\nu\rho} + \Gamma^\sigma_{\sigma\kappa}\Gamma^\kappa_{\nu\rho}\right) - \left(\partial_\nu \Gamma^\sigma_{\sigma\rho} + \Gamma^\sigma_{\nu\kappa}\Gamma^\kappa_{\sigma\rho}\right)$$

HW
$$\begin{cases} \Gamma^t_{tr} = \dfrac{A'}{2A} \quad \Gamma^r_{tt} = \dfrac{A'}{2B} \quad \Gamma^r_{rr} = \dfrac{-B'}{2B} \quad \Gamma^r_{\theta\theta} = \dfrac{-r}{B} \quad \Gamma^r_{\varphi\varphi} = \dfrac{-r\sin^2\theta}{2B} \\ \Gamma^\theta_{r\theta} = \dfrac{1}{r} \quad \Gamma^\varphi_{r\varphi} = \dfrac{1}{r} \quad \Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta \quad \Gamma^\varphi_{\theta\varphi} = \cot\theta \end{cases}$$

os outros = 0

$$\left\{\begin{aligned}
R_{tt} &= \cdots = \frac{A''}{2B} + \frac{A'}{4B} - \frac{A'}{4B}\left(\frac{A'}{A}+\frac{B'}{B}\right) = 0 \\
R_{rr} &= \cdots = -\frac{A''}{2A} + \frac{B'}{2B}(\ldots) = 0 \\
R_{\theta\theta} &= \cdots = 0 \\
R_{\varphi\varphi} &= r^2 R_{\theta\theta} = \cdots = 0
\end{aligned}\right.$$

4W

3 equações de 2ª ordem para 2 funções

$A(r)\ B(r)$

Será que o sistema é sobredeterminado?

Não $\longrightarrow$ identidade de Bianchi (ver depois)

$$\frac{R_{tt}}{A} + \frac{R_{rr}}{B} = \frac{1}{rB}\underbrace{\left(\frac{A'}{A}+\frac{B'}{B}\right)}_{=0} = 0$$

$$\frac{d}{dr}\ln A + \frac{d}{dr}\ln B = 0 \rightarrow \frac{d}{dr}\ln(AB) = 0$$

$$\rightarrow A\cdot B = cte = 1$$

↳ condição ∞

$$\boxed{B = \frac{1}{A}}$$

$$\rightarrow r\left(\frac{1}{B}\right)' + \frac{1}{B} = 1 \qquad \frac{1}{B} = 1 + \frac{cte}{r} = A$$

$A = \frac{1}{B}$

$$\boxed{cte = -2GM}$$

$$\boxed{\begin{aligned} ds^2 &= -A(r)\,dt^2 + B(r)\,dr^2 + r^2\, d\Omega_2^2 \\ &= -\left(1-\frac{2GM}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2GM}{r}} + r^2\, d\Omega_2^2 \end{aligned}}$$

$d\theta^2 + \sin^2\theta\, d\varphi^2$

Métrica de Schwarzschild — válida para $\boxed{r > R}$

raio da estrela

$\boxed{r_s = 2GM}$ → singularidade de coordenadas!

Singularidade na métrica!

$$\begin{cases} g_{tt} \to 0 \\ g_{rr} \to \infty \end{cases}$$

Singularidade de curvatura vs. singularidade de coordenadas

Escalar de curvatura $\underbrace{R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}}_{\text{Kretschmann}} = \left.\frac{12(2GM)^2}{r^6}\right|_{r=r_s}$ finito!

$(c \neq 1)$

$$r_s = \frac{2GM}{c^2}$$

$$\begin{cases} M_{sun} = 2\times 10^{30}\,\text{kg} \\ M_{Earth} = 6\cdot 10^{24}\,\text{kg} \end{cases}$$

$r_s|_{Earth} = \dots = 1\,\text{cm}$

$G = 6.7\times 10^{-11}\,\frac{m^3}{kg\, s^2}$

$r_s|_{sun} = \dots = 3\,\text{km}$

$c = 3\cdot 10^8\,\text{m/s}$

Constante cosmológica

$$\int d^4x \sqrt{-g} \times \text{escalar}$$

↓

pode ser um número puro

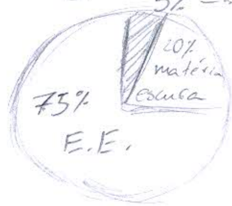
constante cosmológica

$$S_{EH} + S_{cosmo} = \frac{1}{16\pi G} \int d^4x \sqrt{-g}\,(R - 2\lambda)$$

$$\delta(S_{EH} + S_{cosmo}) = 0$$

$$0 = \delta(\sqrt{-g})(R - 2\Lambda) + \sqrt{-g}\,\delta R$$

↳ cte cosmológica → Energia Escura



HW

$$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0}$$

uma const. muda as eq. do movimento!

devido à $\sqrt{-g}$