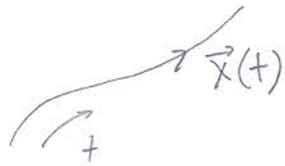
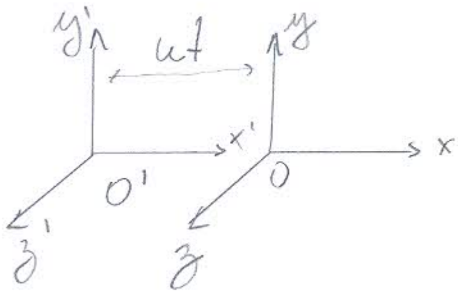
17/02/16 - Relatividade

* Mecânica de Newton



$$\boxed{m\frac{d^2\vec{x}}{dt^2} = \vec{F}}$$

1) Invariante sobre transformações de Galileo



$u = \text{const.}$

$\boxed{t = t'}$ relógios universais

$$\begin{cases} x' = x + ut \\ y' = y \\ z' = z \end{cases}$$

$$\frac{dx'}{dt'} = \frac{dx'}{dt} = \frac{dx}{dt} + u \longrightarrow \boxed{v' = v + u}$$

$$\frac{d^2x'}{dt'^2} = \frac{d^2x}{dt^2} + 0 \longrightarrow a' = a$$

Eletromagnetismo não é invariante sob transformações de Galileo.

$u \neq \text{const.}$ $\quad ma' = ma + m\frac{du}{dt}$ → Força fictícia

2) 2 derivadas (Dinâmica)

3) Conservação

$\vec{F} = \vec{F}(\vec{x})$ mas não de $\dot{\vec{x}}$

$$F^i(\vec{x}) = -\frac{\partial V(\vec{x})}{\partial x^i} = m\frac{d^2}{dt^2}x^i \quad \Big| \; \frac{dx^i}{dt}$$

$$\underbrace{m\frac{d^2x^i}{dt^2}\frac{dx^i}{dt}}_{\frac{1}{2}\frac{d}{dt}\left(\frac{dx^i}{dt}\right)^2} = -\frac{dx^i}{dt}\frac{\partial V}{\partial x^i} \quad \Big| \; \sum_i$$

$$\Rightarrow \frac{d}{dt}\underbrace{\left(\frac{m}{2}\sum_i\left(\frac{dx^i}{dt}\right)^2 + V(\vec{x})\right)}_{\text{const} \equiv E} = 0$$

F^i = central

$$V(\vec{x}) = V(r) \qquad r = \sqrt{\vec{x}\cdot\vec{x}} = \sqrt{\sum_i (x^i)^2}$$

$$\frac{\partial}{\partial x^i}V(r) = \frac{\partial r}{\partial x_i}V'(r) = \frac{x^i}{r}V'(r)$$

$$x^j \; \Big| \; F^i = -\frac{V'(r)}{r}x^i = m\frac{d^2}{dt^2}x^i$$

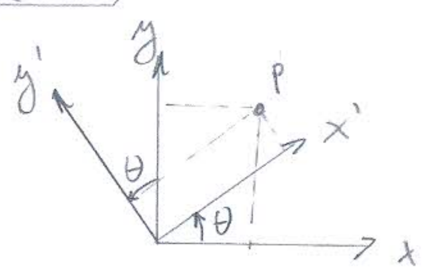
$$\left(m\frac{d^2x^i}{dt^2}x^j = -\frac{V'}{r}x^ix^j\right) - (i \leftrightarrow j)$$

$$m\frac{d^2x^i}{dt^2}x^j - m\frac{d^2x^j}{dt^2}x^i = -\frac{V'}{r}x^ix^j + \frac{V'}{r}x^jx^i$$

$$m\frac{d}{dt}\underbrace{\left(x^j\frac{dx^i}{dt} - x^i\frac{dx^j}{dt}\right)}_{= \text{const} = \ell^{ij}} = 0$$

$\rightarrow$ Teorema de Nether : Simetria $\leftrightarrow$ Conservação

$\boxed{\text{Rotações}}$ 2D $\rightarrow$ deixa invariante o comp. de um vetor



(transf. passiva)

$$x' = \cos\theta\, x + \sin\theta\, y$$
$$y' = -\sin\theta\, x + \sin\theta\, y$$

Vetor

$$\underbrace{\begin{pmatrix} x' \\ y' \end{pmatrix}}_{\vec{r}\,'} = \underbrace{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}}_{R(\theta)} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\vec{r}} \qquad \vec{r}\,' = R(\theta)\,\vec{r}$$

compumento de um vetor $\vec{p}^{\,+} \cdot \vec{p}$

$$\vec{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \qquad \vec{p}^{\,T} = (p_1 , p_2) \qquad \vec{p}^{\,T} \cdot \vec{p} = (p_1)^2 + (p_2)^2 = \vec{p}^{\,2}$$

$$\vec{p} = \vec{u} + \vec{v} \qquad \vec{p}^{\,2} = c^{\underline{te}} \text{ sobre rotações}$$
$$= (\vec{u} + \vec{v})^T \cdot (\vec{u} + \vec{v}) = \vec{u}^{\,2} + \vec{v}^{\,2} + 2\vec{u}^{\,T}\vec{v}$$
$$\longleftrightarrow \underbrace{\vec{u}\,'^2}_{\vec{u}^{\,2}} + \underbrace{\vec{v}\,'^2}_{\vec{v}^{\,2}} + 2\vec{u}\,'^T \cdot \vec{v}\,' = \vec{p}\,'^2$$

Deve ser $\vec{u}^{\,T} \cdot \vec{v} = \vec{u}\,'^T \cdot \vec{v}\,'$

$$\vec{u}\,'^T \cdot \vec{v}\,' = \vec{u}^{\,T} R^T(\theta)\, R(\theta)\, \vec{v} = \vec{u}^{\,T} \cdot \vec{v}$$
$$= \vec{u}^{\,T} \cdot \mathbb{1} \cdot \vec{v}$$

$\boxed{R^T R = \mathbb{1}}$ Ortogonal

$$R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \qquad R^T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = R^{-1}$$

$$\det(R^T R) = \det \mathbb{1} = 1$$
$$\shortparallel$$
$$\det(R^T)\det(R)$$
$$\shortparallel$$
$$(\det(R))^2 = 1 \Rightarrow \det(R) = \pm 1$$

$O(D)$ $(O(2))$ $\rightarrow$ conjunto de transf. ortogonais

$SO(D) \rightarrow \boxed{\det = +1}$

$$O(D) = \begin{cases} R^T R = \mathbb{1} \\ \det R = \pm 1 \end{cases} \qquad SO(D) = \begin{cases} R^T R = \mathbb{1} \\ \det R = +1 \end{cases}$$

Exemplo. $P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ "inversão de paridade"

$$\det P = -1$$
$$P^T = P^{-1}$$

Rotações infinitesimais (Lie)

$$R(\theta) \simeq \mathbb{1} + A + \mathcal{O}(A^2) \qquad \underline{SO(D)}$$

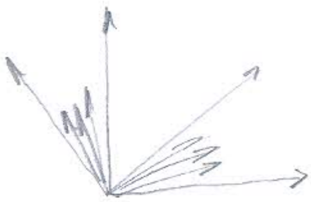
$$R^T R = \mathbb{1} \qquad (\mathbb{1} + A)^T \cdot (\mathbb{1} + A) = \mathbb{1} + A^T + A + \underbrace{\cancel{A^T A}}_{}^{2^a\text{ ordem}}$$

$\Rightarrow A^T = A$ $\quad$ A = antissimetrica

$$D = 2 \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad A = \theta J = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}$$

$$R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & \theta \\ -\theta & 1 \end{pmatrix} = \mathbb{1} + \theta J$$

$$J = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{\text{gerador da transformação}}$$



$$e^x = \lim_{N\to\infty}\left(1+\frac{x}{N}\right)^N$$

$$\lim_{N\to\infty}\left(1+\frac{\theta J}{N}\right)^N = e^{\theta J} = R(\theta)$$

$$R(\theta) = \lim_{N\to\infty}\left(R\left(\frac{\theta}{N}\right)\right)^N$$

Rotação finita θ

Rotação infinitesimal θ/N

$$\overbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}^{J}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \overbrace{\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}}^{-\mathbb{1}}$$

$$e^{\theta J} = \sum_{m=0}^{\infty}\frac{\theta^m J^m}{m!} = \sum_{k=0}^{\infty}\frac{\theta^{2k}\overbrace{J^{2k}}^{(-1)^k}}{(2k)!} + \sum_{k=0}^{\infty}\frac{\theta^{2k+1}\overbrace{J^{2k+1}}^{(-1)^k J}}{(2k+1)!}$$

$$= \cos\theta\,.\,\mathbb{1} + \sin\theta\, J$$

Até agora a gente escolheu a O (= origem) mas não é necessário

(P, Q) $\quad \vec{r}_p = (x, y) \quad \vec{r}_q = (\tilde{x}, \tilde{y})$

$$\vec{r}_p{}' = (x', y') = R(\theta)\vec{r}_p \qquad \vec{r}_q{}' = (\tilde{x}', \tilde{y}') = R(\theta)\vec{r}_Q$$

$$\vec{r}_Q - \vec{r}_p \longrightarrow R(\theta)(\vec{r}_Q - \vec{r}_p)$$

$$\begin{pmatrix} \Delta x' \\ \Delta y' \end{pmatrix} = R(\theta)\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \qquad \begin{aligned} \Delta x &= \tilde{x} - x \\ \Delta y &= \tilde{y} - y \end{aligned}$$

$$\boxed{\begin{pmatrix} dx' \\ dy' \end{pmatrix} = R(\theta)\begin{pmatrix} dx \\ dy \end{pmatrix}}$$

$\vec{d_x}$ é o protótipo de um vetor

Exemplo de não-vetor

$\vec{P}$ vetor $\vec{P} = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$ $\begin{pmatrix} P_1' \\ P_2' \end{pmatrix} = R(\theta)\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$

$\begin{pmatrix} aP_1 \\ bP_2 \end{pmatrix}$ $R(\theta)\begin{pmatrix} a\vec{P_1} \\ b\vec{P_2} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} aP_1 \\ bP_2 \end{pmatrix}$

$$= \begin{pmatrix} \cos\theta\, aP_1 + \sin\theta\, bP_2 \\ -\sin\theta\, aP_1 + \cos\theta\, bP_2 \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} aP_1' \\ bP_2' \end{pmatrix}$$

$a\vec{P_1}' = a(\cos\theta\, P_1 + \sin\theta\, P_2)$ é um vetor apenas

$bP_2' = b(-\sin\theta\, P_1 + \cos\theta\, P_2)$ quando $a = b$

SO(3)

$$R = \begin{pmatrix} \circ & \circ & \circ \\ \circ & \circ & \circ \\ \circ & \circ & \circ \end{pmatrix} = \mathbb{1} + A$$

$A^T = -A$

$$= \theta_x J_x + \theta_y J_y + \theta_z J_z$$

$J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$; $J_y = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$; $J_z = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\vec{P} = \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix}$; $\vec{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$ | São vetores de SO(3)

$\begin{pmatrix} p^2 q^3 \\ p^3 q^1 \\ p^1 q^2 \end{pmatrix}$ é um vetor?
(Não)

$$\begin{pmatrix} p^2 q^3 - p^3 q^2 \\ p^3 q^1 - p^1 q^3 \\ p^1 q^2 - p^2 p^1 \end{pmatrix}$$ é um vetor?

(Sim)