24/02/16 - Relatividade

T^{ij} D^2 componentes

$i,j = 1, \dots, D$

→ parte anti-simétrica

$$T^{ij} = \begin{cases} A^{ij} = (T^{ij} - T^{ji}) & D(D-1)/2 \text{ componentes} \\ S = S^{ii} \left(= \sum_{i=1}^{D} S^{ii}\right) & 1 \text{ componente} \\ \tilde{S}^{ij} = \underbrace{(T^{ij} + T^{ji})}_{S^{ij}} - \frac{\delta_{ij} S}{D} & \frac{D(D+1)}{2} - 1 \end{cases}$$

traço ←

Parte simétrica sem o traço ←

$$\sum_{i=1}^{D} \tilde{S}^{ii} = 0$$

Convenção de Einstein

$\tilde{S}^{ij}$ é sem traço $\quad \sum_{i,j} \delta^{ij} \Big| \tilde{S}^{ij} = \sum_{i} \tilde{S}^{ii} = \tilde{S}^{ii}$

$$\sum_{i,j} \delta^{ij} \Big| \left(S^{ij} - \frac{\delta^{ij} S}{D}\right) = \underset{S}{\underset{\shortparallel}{S^{ii}}} - \frac{\delta^{ij}\delta^{ij}}{D} = S - \frac{D}{D} S = 0$$

$$D^2 = \frac{D(D-1)}{2} \oplus 1 \oplus \left(\frac{D(D+1)}{2} - 1\right)$$

$$D = 3 : \quad 9 = 3 \oplus 1 \oplus 5$$

$$D(R) = \begin{pmatrix} 3\times 3 & & \\ & 1\times 1 & \\ & & 5\times 5 \end{pmatrix}_{9\times 9}$$

Subgrupos

$$SO(2) \subset SO(3)$$

$$\begin{array}{cccc} ① & ② & ③ & \\ & SO(2) & & \\ \cos\theta & \sin\theta & 0 & ① \\ -\sin\theta & \cos\theta & 0 & ② \\ 0 & 0 & 1 & ③ \\ & SO(3) & & \end{array}$$

$\boxed{V^i \quad i=1,2,3}$ Vetor de $SO(3)$

$\underbrace{(V^1, V^2)}_{\text{vetor sob } SO(2)}$ V^3 Não se transforma sob $SO(2)$ → (Singoleto ou Escalar)

T^{ij}

duplet = dupleto

HW

$A^{ij} \to 3 = 2 + 1$

Singoletos = parte de T^{ij} que não se transformam sob $SO(2)$

$\tilde{S}^{ij} \to 5 = 2 \oplus 2 \oplus 1$

dois dupletos

$\tilde{S}^{ij}$: $\tilde{S}^{11}$, $\tilde{S}^{12}$, $\tilde{S}^{13}$, $\tilde{S}^{22}$, $\tilde{S}^{23}$

$(\tilde{S}^{11} + \tilde{S}^{22} + \tilde{S}^{33} = 0)$ → condição de S não ter traço

$\Rightarrow \tilde{S}^{33}$ não é independente.

$SO(2)$

$\downarrow$

$$V^3 \longrightarrow (V^3)' = V^3 \Rightarrow \begin{cases} R^{33} = 1 \\ R^{13} = R^{23} = R^{31} = R^{32} = 0 \end{cases}$$

é o singoleto

$\Rightarrow \tilde{S}^{11} + \tilde{S}^{22} \; (= -\tilde{S}^{33})$ não se transforma sobre $SO(2)$

HW

Os dois dupletos são :

$(\tilde{S}^{13}, \tilde{S}^{23})$ (<u>HW</u>)

$(\tilde{S}^{12}, \tilde{S}^{11} - \tilde{S}^{11} - \tilde{S}^{22})$

Tensores invariantes

δ^{ij} $\quad \epsilon^{i_1 i_2 \dots i_D}$ (ex: $D=2$ $\epsilon^{12} = -\epsilon^{21} = 1$

$D=3$ $\epsilon^{123} = \epsilon^{231} = \dots = 1$

$\vdots$)

*) δ^{ij} $\quad R^T R = \mathbb{1}$

$$\delta^{ij} \longrightarrow (\delta^{ij})' = R^{ik} R^{jl} \delta^{kl}$$

$$= R^{ik} R^{jk} = R^{jk} R^{ik}$$

$$= R^{jk} (R^T)^{ki} = \delta^{ji} = \delta^{ij}$$

*) $\epsilon^{\overbrace{ijk\dots m}^{D\text{-indices}}}$

$(\det R = 1) \times \epsilon^{pqr\dots s}$

$$\epsilon^{pqr\dots s} \underbrace{\det R}_{\epsilon^{ijk\dots l} R^{i1} R^{j2} \dots R^{lD}} = \epsilon^{pqr\dots s}$$

(HW) $\overbrace{\epsilon^{i,j,k,\dots l} R^{ip} R^{jq} \dots R^{ls}}$

$$\begin{vmatrix} R^{11} & R^{12} \\ R^{21} & R^{22} \end{vmatrix} = R^{11} R^{22} - R^{12} R^{21}$$

$$\epsilon^{ij} R^{i1} R^{j2} = \overset{+1}{\epsilon^{12}} R^{11} R^{22} + \underset{-1}{\epsilon^{21}} R^{21} R^{12}$$

$D=2 \quad \epsilon^{ij} R^{ip} R^{jq} = \epsilon^{pq}$ p, q livres

$\left.\begin{matrix} p=1 \\ q=1 \end{matrix}\right\} \quad \epsilon^{ij} R^{i1} R^{j1} = 0$

$$\underset{\overset{\shortparallel}{1}}{\epsilon^{12}} R^{11} R^{21} + \underset{-1}{\epsilon^{21}} R^{21} R^{11} = 0$$

$\left.\begin{matrix} p=1 \\ q=2 \end{matrix}\right\} \quad$ (HW)

(*) $\sum_{k} \epsilon^{ijk} \epsilon^{lmk} = \delta^{il}\delta^{jm} - \delta^{im}\delta^{jl}$ (HW)

↳ Explica todas as relações $\vec{A}\cdot(\vec{B}\times\vec{C}) = \dots$

(*) $\underset{\shortparallel}{S^{ij} A^{ij}} = 0$

$S^{ji} A^{ji} = S^{ij}(-A^{ij})$

$(\vec{B}\times\vec{C})_i = \epsilon_{ijk} B_j C_k \qquad (\vec{B}\times\vec{C})\cdot(\vec{D}\times\vec{E}) = \epsilon_{ijk}(B_j C_k)\epsilon_{ilm}(D_l E_m)$

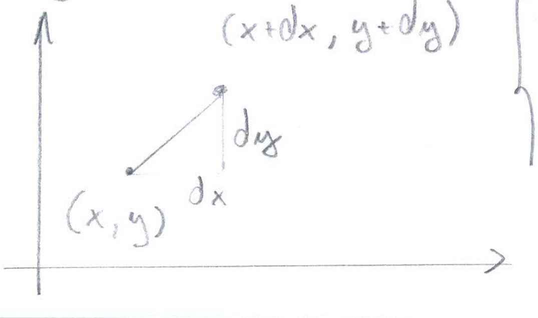
$(D\times E)_i = \epsilon_{ilm} D_l E_m$

Troca

(Câmbio) de coordenadas

Espaço Euclidiano 2D

① (x, y)



$\left\{\begin{matrix} P = (x, y) \\ Q = (x+dx, y+dx) \end{matrix}\right.$

Coordenadas Cartesianas

$\underbrace{ds}_{\text{distância}}^2 = dx^2 + dy^2$ $\qquad$ $ds^2 \equiv$ elemento de linha

② **Coordenadas polares** (r, θ)

$(x, y) \longrightarrow (r, \theta)$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \longrightarrow \begin{array}{l} dx = \cos\theta\, dr - r\sin\theta\, d\theta \\ dy = \sin\theta\, dr + r\cos\theta\, d\theta \end{array}$$

$$ds^2 = dx^2 + dy^2 = dr^2 + r^2 d\theta^2$$

③ **Coord. genéricas**

$(x, y) \rightarrow (u, v)$ $\qquad$ $dx = f_u du + f_v dv$

$\begin{cases} x = f(u, v) \\ y = g(u, v) \end{cases}$ $\qquad$ $dy = g_u du + g_v dv$

$$dx^2 + dy^2 = (f_u^2 + g_u^2)du^2 + (f_v^2 + g_v^2)dv^2 + $$
$$+ 2(f_u f_v + g_u g_v)du\,dv$$

3D $(x, y, z) \longrightarrow (r, \theta, \varphi)$

$$\begin{array}{l} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{array} \longrightarrow \boxed{\begin{array}{l} ds^2 = dx^2 + dy^2 + dz^2 \\ \quad = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\varphi^2 \end{array}}$$

Para conveniência futura, vamos acostumar com índices "gregos".

$x^\mu = (x^1, x^2, \ldots, x^D)$ Coord. Cartesianas

$$ds^2 = \sum_{\nu=1}^{D} (dx^\nu)^2 = dx^1 dx^1 + dx^2 dx^2 + \ldots + dx^D dx^D$$

$$= \sum_{\mu=1}^{D} \sum_{\nu=1}^{D} g_{\mu\nu}\, dx^\mu dx^\nu$$

$$\underbrace{g_{\mu\nu} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & 1 \end{pmatrix}_{D\times D}}_{\text{Métrica}} = \delta_{\mu\nu}$$

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

Notação

$$\sum_{\mu=1}^{D} (dx^\mu)^2 = \sum_{\nu=1}^{D} dx^\nu dx^\nu$$

$$= (dx^\mu)^2$$

$$dx^\nu = (dr, d\theta, d\varphi)$$

Aqui não faz diferença dx^μ ou dx_μ, mas vamos nos acostumar a escrever índices contraídos um em cima e o outro em baixo

Coord Esféricas (3D)

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2$$

$x^\nu = (r, \theta, \varphi)$ $\quad x^r, x^\theta, x^\varphi$

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

$$\begin{cases} g_{rr} = 1 & g_{r\theta} = g_{r\varphi} = 0 & g_{\theta r} = g_{\varphi r} = 0 \\ g_{\theta\theta} = r^2 & g_{\theta\varphi} = 0 & g_{\varphi\theta} = 0 \\ g_{\varphi\varphi} = r^2 \sin^2\theta & & \end{cases}$$

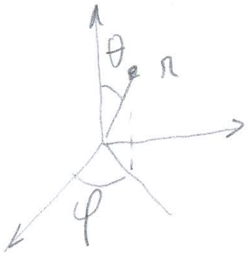
$dx^\mu dx^\nu = dx^\nu dx^\mu$

$\boxed{g_{\mu\nu} = g_{\nu\mu}}$ Simétrico

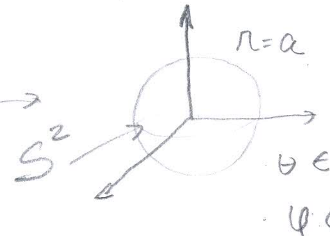
$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin\theta \end{pmatrix}$$

$g_{\mu\nu}(x)$ em geral depende das coordenadas

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2 \longrightarrow E^3 = \mathbb{R}^3$$



$r = a \longrightarrow$



$\theta \in [0, \pi]$

$\varphi \in [0, 2\pi]$

$$ds^2 = a^2 d\theta^2 + a^2 \sin^2\theta \, d\varphi^2$$
$$= \underbrace{a^2 (d\theta^2 + \sin^2\theta \, d\varphi^2)}_{\text{métrica de 2-esfera } (S^2)}$$

* S^2 está embutada em $\mathbb{R}^3$

$$\underbrace{g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix}}_{\mathbb{R}^3}$$

Dado um espaço com métrica $g_{\mu\nu}(x)$, como posso saber se o espaço é plano (como $\mathbb{R}^3$) ou curvo (como S^2)?

$$ds^2 = (1 + u^2) du^2 + (1 + \textcircled{8} v^2) dv^2 + 2(2v - u) du dv$$

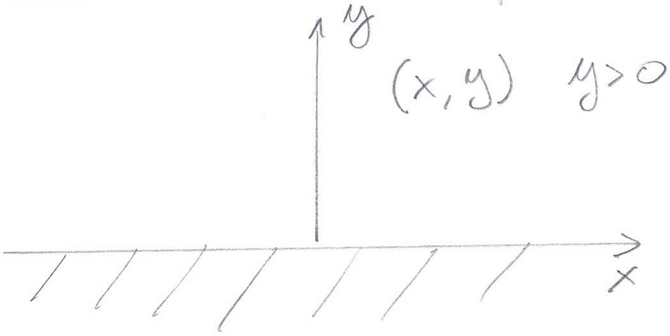
$$ds^2 = (1 + u^2) du^2 + (1 + \textcircled{2} v^2) dv^2 + 2(2v - u) du dv$$

<u>Obs</u>: Abuso de Notação. É comum dizer que a métrica é igual ao elemento de linha

→ Curvatura : A curvatura está embutida na métrica

↳ Gauss/Riemann

Exemplo: Semi-plano de Poincaré



$$ds^2 = \frac{dx^2+dy^2}{y^2}$$

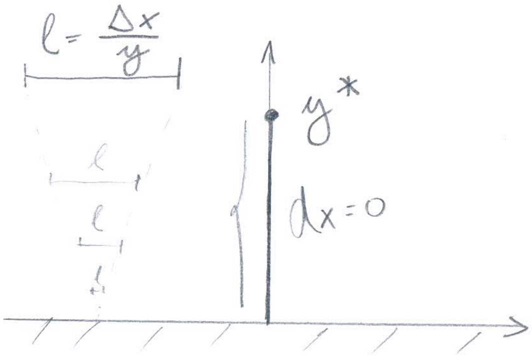
↓

Invariante sob translações em x.

$$\left.\begin{array}{l} x_1 \longrightarrow \tilde{x}_1 = x_1 + \Delta x \\ x_2 \longrightarrow \tilde{x}_2 = x_2 + \Delta x \end{array}\right] \quad dx = x_1 - x_2 = \tilde{x}_1 - \tilde{x}_2$$

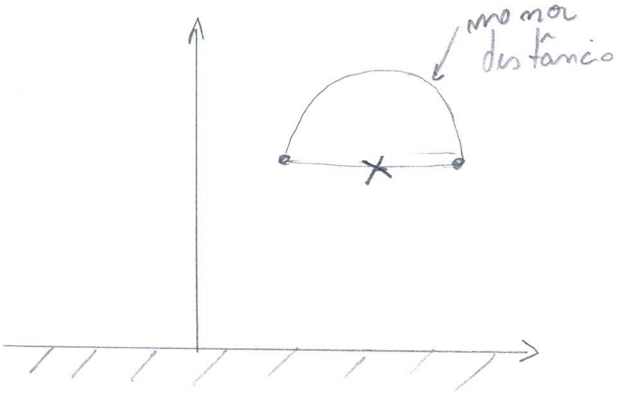
Mas não é invariante sób $y \longrightarrow y + \Delta y$

$$\boxed{\begin{array}{l} g_{\mu\nu}(x+a, y) = g_{\mu\nu}(x, y) \\ g_{\mu\nu}(x, y+a) \neq g_{\mu\nu}(x, y) \end{array}}$$

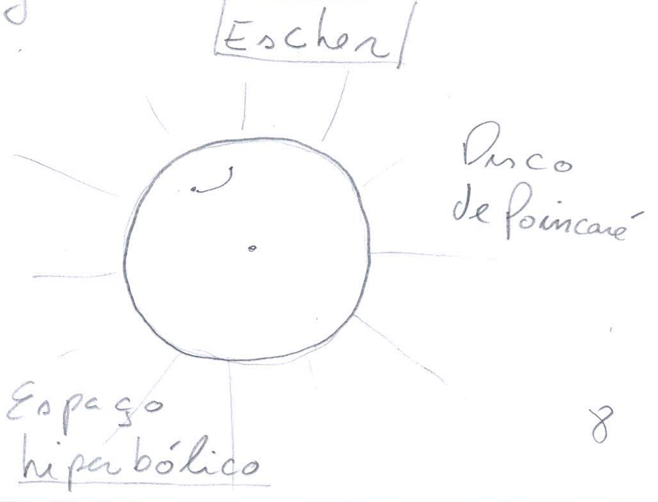


$$\int ds = \int_{0^+}^{y^*} \frac{dy}{y} = \log\left(\frac{y^*}{0^+}\right) \xrightarrow[0^+\to 0]{} \infty$$

$$\Delta x = \ell y \xrightarrow[y \to 0]{} 0$$



$$\xrightarrow[\text{disco}]{\text{Plano}}$$



Exemplo: Espaço plano em coordenadas de Boyer-Lindquist (→ Kerr)

$$\begin{cases} x = f(r)\sin\theta\cos\varphi \\ y = f(r)\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases} \longrightarrow ds^2 = dx^2 + dy^2 + dz^2$$

$$ds^2 = (f'^2\sin^2\theta + \cos^2\theta)\,dr^2 + (f'^2\cos^2\theta + r^2\sin^2\theta)\,d\theta^2 + f^2\sin^2\theta\,d\varphi^2 + 2\underbrace{(ff' - r)}_{=0}\sin\theta\cos\theta\,dr\,d\theta \qquad ff' = r$$

$$df \cdot f = r\,dr$$

$$\boxed{f^2 = r^2 + \text{const} = r^2 + a^2} \longrightarrow$$ Escolha de f que garante a métrica diagonal

↳ $a = 0 \Rightarrow$ coord. esféricas

$$a \neq 0 \quad \boxed{ds^2 = \frac{r^2 + a^2\cos^2\theta}{r^2 + a^2}\,dr^2 + (r^2 + a^2\cos^2\theta)\,d\theta^2 + (r^2 + a^2)\sin^2\theta\,d\varphi^2}$$

$r = \text{Const.} \rightarrow ds^2 =$ elipsóide (não uma esfera)

$r = 0 \longrightarrow$ não é um ponto, mas um disco de raio a.

$$ds^2 = a^2\cos^2\theta + a^2\sin^2\theta\,d\varphi^2$$