Relatividade - 29/02

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$ - Poincaré

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2$$ - $\mathbb{R}^3$

$r \longrightarrow r + a$

Não é evidente a partir da métrica

Vetores de Killing: Isometrias (transf. que deixam a métrica invariante)

Transformações gerais de Coordenadas

$$x^\mu \longrightarrow x'^\mu$$

ex: $(x, y) \longrightarrow (r, \theta)$

$$\underbrace{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}_{\text{Não lineares}}$$

mas as transformações infinitesimais são lineares

$$dx = \cos\theta \, \underline{dr} - \sin\theta \, r \, \underline{d\theta}$$
$$dy = \sin\theta \, \underline{dr} - \cos\theta \, r \, \underline{d\theta}$$

$$\boxed{dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu}$$

$$\frac{\partial}{\partial x^\nu} = \partial_\nu$$

$$\boxed{\frac{\partial x'^\mu}{\partial x^\nu} = \partial_\nu x'^\mu \equiv S^\mu_\nu(x)}$$

$$\boxed{dx'^{\mu} = S^{\mu}{}_{\nu}(x)\, dx^{\nu}}$$ Relação linear → vetor sobre transf. geral de coordenadas

$$dx^{\mu} = \frac{\partial x^{\mu}}{\partial x'^{\nu}}\, dx'^{\nu} \equiv (S^{-1})^{\mu}_{\nu}\, dx'^{\nu}$$

$S^{\mu}_{\nu}(x)$ é uma generalização de $dx'^{\mu} = R^{\mu}_{\nu}\, dx^{\nu}$ mas dependente de x!

$$(S^{-1})^{\mu}_{\rho}\, S^{\rho}_{\nu} = \frac{\partial x^{\mu}}{\partial x'^{\rho}} \frac{\partial x'^{\rho}}{\partial x^{\nu}} = \delta^{\mu}_{\nu}$$

Ex: $(x^1, x^2) \longrightarrow (x'^1, x'^2)$

$$dx'^1 = dr = \frac{x\,dx + y\,dy}{\sqrt{x^2+y^2}}$$

$$dx'^2 = d\theta = \frac{x\,dy - y\,dx}{\sqrt{x^2+y^2}}$$

$$S^1_1 = \frac{x}{\sqrt{x^2+y^2}} \qquad S^1_2 = \frac{y}{\sqrt{x^2+y^2}}$$

$$S^2_1 = -\frac{y}{x^2+y^2} \qquad S^2_2 = \frac{x}{\sqrt{x^2+y^2}}$$

E a métrica?

$$ds^2 = g_{\mu\nu}(x)\, dx^{\mu} dx^{\nu} = g'_{\rho\sigma}(x')\, dx'^{\rho} dx'^{\sigma}$$

mesmo ponto p descrito em 2 coordenadas diferentes x^{μ} e x'^{μ}

$$ds^2 = g_{\mu\nu}(x)\frac{\partial x^\mu}{\partial x'^\rho}\frac{\partial x^\nu}{\partial x'^\sigma}dx'^\rho dx'^\sigma$$

$$\boxed{g'_{\rho\sigma}(x') = g_{\mu\nu}(x)(S^{-1})^\mu{}_\rho(S^{-1})^\nu{}_\sigma} \quad (*)$$

$x^\mu \to x'^\mu$

índices altos $\longrightarrow S$

índices baixos $\longrightarrow S^{-1}$

$ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ fica invariante

dx^μ: Vetor

∂_μ: Vetor (dual)

$$\partial_\mu = \frac{\partial}{\partial x^\mu}$$

$$\partial'_\mu = \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu} = (S^{-1})^\nu\partial_\nu$$

Matrizes

$AB = A^{ij}B_{jk}$

$$g'_{\rho\sigma}(x') = g_{\mu\nu}(x)(S^{-1})^\mu{}_\rho(S^{-1})^\nu{}_\sigma$$

$$= (S^{-1})^\mu{}_\rho\, g_{\mu\nu}(x)\,(S^{-1})^\nu{}_\sigma$$

$$= ((S^{-1})^T)_\rho{}^\mu\, g_{\mu\nu}(x)\,(S^{-1})^\nu{}_\sigma$$

$$\boxed{g'(x') = (S^{-1})^T g(x) S^{-1}} \quad (*)$$

Lembrando

$$R^T R = \mathbb{1}$$

$$\mathbb{1} = R^T \mathbb{1} R$$

Uma rotação é uma transformação geral de coordenadas que deixa a métrica euclidiana ($\mathbb{1}$) invariante!

Métrica inversa

$$g^{\mu\nu} g_{\nu\rho} = \delta^{\mu}{}_{\rho}$$

ou $(g^{-1})^{\mu\nu}$ (índices para cima são da métrica inversa)

$$g'^{\mu\nu}(x') = S^{\mu}{}_{\rho} S^{\nu}{}_{\sigma} g^{\rho\sigma}(x) \qquad (HW)$$

Escalares, vetores, tensores sobre T.G.C.

Vetor "contravariante"

$$W'^{\mu}(x') = S^{\mu}{}_{\nu}(x) W^{\nu}(x)$$

Vetor "covariante"

$$W'_{\mu}(x') = W_{\rho}(x) (S^{-1})^{\rho}{}_{\mu}(x)$$

Escalar

$$\phi'(x') = \phi(x)$$

$$V'_\mu(x')\,W^{\mu\,\prime}(x') = V_\rho(x)\underbrace{(S^{-1})^\rho{}_\mu\, S^\mu{}_\nu}_{\delta^\rho{}_\nu} W^\nu(x)$$

$$= V_\rho(x)\, W^\rho(x)$$

$\left(\text{Em geral, } S^{-1} \neq S^T\right)$

Abaixar/Subir índices:

$$\begin{cases} W^\nu \longrightarrow W_\mu = g_{\mu\nu} W^\nu \\ V_\mu \longrightarrow V^\mu = g^{\mu\nu} V_\nu \end{cases}$$

Verificar:

$$W'_\mu = g'_{\rho\sigma} W'^\sigma = g_{\mu\nu}(S^{-1})^\mu{}_\rho \overbrace{(S^{-1})^\nu{}_\sigma\, S^\sigma{}_\lambda}^{\delta^\nu{}_\lambda} W^\lambda$$

$$= g_{\mu\nu}(S^{-1})^\mu{}_\rho W^\nu = (S^{-1})^\mu{}_\rho W_\mu \quad \square$$

Ex: Resolver

$$g_{\mu\rho} A^\rho = B_\mu \qquad A^\rho = ?$$

$$\sum_\mu g^{\sigma\mu}\left(g_{\mu\rho} A^\rho = B_\mu\right) \Rightarrow \underbrace{\delta^\sigma{}_\rho A^\rho}_{A^\sigma} = g^{\sigma\mu} B_\mu$$

$$\sigma \to \rho$$

$$\boxed{A^\rho = g^{\rho\mu} B_\mu}$$

Área e Volume

$d^D x$ não se transforma apropriadamente sobre uma T.G.C. : $x^\mu \longrightarrow x'^\mu$

ex. $dx\,dy\,dz \neq dr\,d\theta\,d\varphi$

$\longrightarrow$ Precisamos de um det. Jacobiano

$$\boxed{d^D x = d^D x' \, J} \quad (**) \qquad J = \det\left(\frac{\partial x^\mu}{\partial x'^\rho}\right)$$

$$\det\left[g'_{\rho\sigma}(x') = g_{\mu\nu}(x)\,\frac{\partial x^\mu}{\partial x'^\rho}\,\frac{\partial x^\nu}{\partial x'^\sigma}\right]$$

$$\det g'_{\rho\sigma}(x') \equiv g' = \underbrace{\det(g_{\mu\nu}(x))}_{\equiv g}\cdot J \cdot J$$

$$g' = g J^2 \; (*) \Longrightarrow \boxed{d^D x \sqrt{g}} = \underbrace{d^D x' J}_{d^D x} \; \underbrace{\frac{\sqrt{g'}}{J}}_{\sqrt{g}}$$

elemento Próprio de Volume

$$\boxed{= d^D x' \sqrt{g'}}$$

$$\underbrace{dx\,dy\,dz}\; = (\;\;)\,dr\,d\theta\,d\varphi$$

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad g_{\mu\nu} = \begin{pmatrix} 1 & & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin\theta \end{pmatrix}$$

$$\Rightarrow g = 1 \qquad\qquad \Rightarrow g' = r^4 \sin^2\theta$$

$$dx\,dy\,dz\,\sqrt{1} = dr\,d\theta\,d\varphi\,\sqrt{r^4 \sin^2\theta} = r^2 \sin\theta\, dr\,d\theta\,d\varphi$$

Divergência, Laplaciano

$\vec{\nabla}\cdot\vec{V} \qquad \vec{\nabla}^2\phi$

Dependem das coord. usadas.

*) $\partial_\mu W^\mu$ não é uma divergência pois não é escalar.

$$\partial_\mu W^\mu \longrightarrow \partial'_\mu W'^\mu = \underbrace{(S^{-1})^\nu{}_\mu\, \partial_\nu}_{\partial'_\mu} \underbrace{(S^\mu{}_\rho W^\rho)}_{=W'^\mu}$$

$$= (S^{-1})^\nu{}_\mu (\partial_\nu S^\mu{}_\rho) W^\rho + $$
$$+ \underbrace{(S^{-1})^\nu{}_\mu S^\mu{}_\rho}_{\delta^\nu_\rho} \partial_\nu W^\rho$$

extra

$$= \partial_\nu W^\nu + \underbrace{(S^{-1})^\nu{}_\mu (\partial_\nu S^\mu{}_\rho) W^\rho}_{}$$

$\neq 0$ para T.G.C.

Truque

$$I = \int \underbrace{d^D x \sqrt{g}}_{\text{escalar}}\; \underbrace{W^\mu(x)\,\partial_\mu\phi(x)}_{\text{escalar}}$$

$\downarrow$ Escalar

(*) A derivada atuando sobre um escalar é um vetor! (HW)

$$I = -\int d^D x\, \partial_\mu(\sqrt{g}\, W^\mu(x))\,\phi(x) + \underbrace{\int d^D x\, \partial_\mu(\sqrt{g}\, W^\mu(x)\phi(x))}_{}$$

stokes

$\boxed{\xrightarrow[x\to\infty]{} 0}$

$$= \int d^{D-1} \sqrt{g}\, W^\mu(x)\,\phi(x)$$

borda: $|x| \to \infty$ $\quad \begin{pmatrix} W^\mu \to 0 \\ \phi \to 0 \end{pmatrix}$ termos de superfície

$$I = -\int \underbrace{d^D x \sqrt{g}}_{\text{escalar}} \underbrace{\frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g}\, W^\mu)}_{\text{deve ser também um escalar}} \underbrace{\phi(x)}_{\text{escalar}}$$

$\cancel{\partial_\mu W^\mu}$

$$\boxed{\frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g}\, W^\mu) \equiv D_\mu W^\mu}$$

Generalização do divergente em espaços curvos.

ex.: coord. esf.

$\sqrt{g} = r^2 \sin\theta$

$$D_\mu W^\mu = \partial_\mu W^\mu + \frac{1}{\sqrt{g}} (\partial_\mu \sqrt{g}) W^\mu$$

$\mu = r, \theta, \varphi$

$$= \partial_r W^r + \partial_\theta W^\theta + \partial_\varphi W^\varphi + \frac{2}{r} W^r + \frac{\cos\theta}{\sin\theta} W^\theta$$

*) Laplaciano:

Truque

$$\int d^D x \sqrt{g} \overbrace{g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi}^{\text{escalar (HW)}}$$

por partes

$$= -\int d^D x \left(\frac{\sqrt{g}}{\sqrt{g}}\right) \partial_\mu (\sqrt{g}\, g^{\mu\nu} \partial_\nu \phi)$$

$$\boxed{D^2 \phi \equiv \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g}\, g^{\mu\nu} \partial_\nu \phi)}$$

Ex. coord. esféricas

$$D^2 \phi = \left(\partial_r^2 + \frac{2}{r} \partial_r + \frac{1}{r^2} \partial_\theta^2 + \frac{\cos\theta}{r^2 \sin\theta} \partial_\theta + \frac{1}{r^2 \sin^2\theta} \partial_\varphi^2\right)$$