Relatividade: 06/04/16 | Exercícios + Superfícies em Minkowski |

Lista 2

1) $$\begin{cases} g_{\mu\nu}(x) = \delta_{\mu\nu} + A_{\mu\nu\lambda} x^{\lambda} + \dots & (1) \\ x^{\mu} = x'^{\mu} + L^{\mu}{}_{\nu\lambda} x'^{\nu} x'^{\lambda} + \dots & (2) \end{cases}$$

$$g'_{\rho\sigma} = \frac{\partial x^{\mu}}{\partial x'^{\rho}} \frac{\partial x^{\nu}}{\partial x'^{\sigma}} g_{\mu\nu}(x) \qquad (3)$$

Derivando (2):

$$\frac{\partial x^{\mu}}{\partial x'^{\rho}} = \frac{\partial x'^{\mu}}{\partial x'^{\rho}} + \underline{2} L^{\mu}{}_{\nu\lambda} \frac{\partial x'^{\nu}}{\partial x'^{\rho}} x'^{\lambda} + \dots$$

contribuição da parte simétrica (trocando os índices)

$$= \delta^{\mu}{}_{\rho} + 2 L^{\mu}{}_{\rho\lambda} x'^{\lambda} + \dots$$

$\mu \leftrightarrow \nu$, $\rho \leftrightarrow \sigma$: $$\frac{\partial x^{\nu}}{\partial x'^{\sigma}} = \delta^{\nu}{}_{\sigma} + 2 L^{\nu}{}_{\sigma\lambda} x'^{\lambda} + \dots$$

Em (3)

$$g'_{\rho\sigma}(x') = (\delta^{\mu}{}_{\rho} + 2 L^{\mu}{}_{\rho\lambda} x'^{\lambda} + \dots)(\delta^{\nu}{}_{\sigma} + 2 L^{\nu}{}_{\sigma\alpha} x'^{\alpha} + \dots) g_{\mu\nu}(x)$$

(4) $\rightarrow$ $$= (\delta^{\mu}{}_{\rho} \delta^{\nu}{}_{\sigma} + 2 \delta^{\mu}{}_{\rho} L^{\nu}{}_{\sigma\alpha} x'^{\alpha} + 2 \delta^{\nu}{}_{\sigma} L^{\mu}{}_{\rho\lambda} x'^{\lambda} + \dots) g_{\mu\nu}(x)$$

(2) em (1):

$$g_{\mu\nu}(x) = \delta_{\mu\nu} + A_{\mu\nu\lambda}(x'^{\lambda} + L^{\lambda}{}_{\alpha\beta} x'^{\alpha} x'^{\beta} + \dots) \qquad (5)$$

(5) em (4): $$g'_{\rho\sigma}(x') = (\delta^{\mu}_{\rho} \delta^{\nu}_{\sigma} + 2\delta^{\mu}_{\rho} L^{\nu}{}_{\sigma\alpha} x'^{\alpha} + 2\delta^{\nu}_{\sigma} L^{\mu}{}_{\rho\lambda} x'^{\lambda} + \dots) \times$$
$$\times (\delta_{\mu\nu} + A_{\mu\nu\tau}(x'^{\tau} + L^{\tau}{}_{\xi\beta} x'^{\xi} x'^{\beta}))$$

Ordem $(x')^0$:

$$\delta^\mu_\rho \delta^\nu_\sigma \delta_{\mu\nu} = \delta^\mu_\rho \delta^\mu_\sigma = \underline{\underline{\delta_{\rho\sigma}}}$$ OK

Ordem $(x')^1$:

$$\delta^\mu_\rho \delta^\nu_\sigma A_{\mu\nu\lambda} x'^\lambda + 2\delta_{\mu\nu}\delta^\nu_\rho L^\nu_{\sigma\lambda} x'^\lambda +$$
$$+ 2\delta_{\mu\nu}\delta^\nu_\sigma L^\mu_{\rho\lambda} x'^\lambda = 0$$

$$x'^\lambda \underbrace{\left(\delta_\rho^\mu \delta^\nu_\sigma A_{\mu\nu\lambda} + 2\delta_{\mu\nu}\delta^\mu{}_\rho L^\lambda{}_{\sigma\lambda} + 2\delta_{\mu\nu}\delta^\nu_\sigma L^\mu{}_{\rho\lambda}\right)}_{=0} = 0$$

$$(*) \quad A_{\rho\sigma\lambda} + \cancel{2\delta_{\rho\nu} L^{\nu\to\mu}_{\sigma\lambda}} + 2\delta_{\mu\sigma} L^\mu_{\rho\lambda} = 0$$

+ ciclicamente
$\rho \to \sigma \to \lambda \to \rho$

$$(**) \quad A_{\sigma\lambda\rho} + 2\delta_{\sigma\mu} L^\mu_{\lambda\rho} + \cancel{2\delta_{\mu\nu} L^\mu_{\sigma\rho}} = 0$$

$$(***) \quad A_{\lambda\rho\sigma} + \cancel{2\delta_{\lambda\mu} L^\mu_{\rho\sigma}} + \cancel{2\delta_{\mu\rho} L^\mu_{\lambda\sigma}} = 0$$

$(*) + (**) - (***)$

$\delta_{\mu\nu} = \delta_{\nu\mu}$
$L^\mu_{\alpha\beta} = L^\mu_{\beta\alpha}$

$$\longrightarrow \boxed{A_{\rho\sigma\lambda} + A_{\sigma\lambda\rho} - A_{\lambda\rho\sigma} = -4\delta_{\mu\sigma} L^\mu_{\lambda\rho}}$$

$\times \delta_{\mu\sigma}$

Lado esquerdo: $-\delta^{\nu\sigma}\delta_{\mu\sigma} L^\mu_{\lambda\rho} = -4L^\nu_{\lambda\rho}$

Lado direito: $A_{\rho\nu\lambda} + A_{\nu\lambda\rho} - A_{\lambda\rho\lambda}$

$$L^{\nu}{}_{\lambda\rho} = \frac{\delta^{\nu\sigma}}{4}\left(A_{\lambda\rho\sigma} - A_{\rho\sigma\lambda} - A_{\sigma\lambda\rho}\right)$$

$B_{\mu\nu,\lambda\sigma}$

Ex. 3) $g_{\mu\nu} = a_\mu \delta_{\mu\nu}$ (Não tem soma sobre μ)

$g_{\mu\mu} \neq 0 \qquad g^{\mu\mu} = \frac{1}{g_{\mu\mu}}$ (Não tem soma sobre μ)

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2g_{\rho\rho}}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\right)$$

<u>4 casos</u>:
$\mu = \nu = \rho$ (i)
$\mu \neq \nu \neq \rho$ (ii)
$\mu = \nu \neq \rho$ (iii)
$\mu \neq \nu = \rho$ (iv)

i) $\Gamma^{\rho}_{\rho\rho} = \frac{1}{2g_{\rho\rho}}\left(\partial_\rho g_{\rho\rho} + \partial_\rho g_{\rho\rho} - \partial_\rho g_{\rho\rho}\right) = \frac{1}{2g_{\rho\rho}}\partial_\rho g_{\rho\rho}$

ii) $\Gamma^{\rho}_{\mu\nu} = 0$

iii) $\Gamma^{\rho}_{\mu\mu} = \cdots = -\frac{1}{2g_{\rho\rho}}\partial_\rho g_{\mu\mu}$

iv) $\Gamma^{\rho}_{\mu\rho} = \cdots = \frac{1}{2g_{\rho\rho}}\partial_\mu g_{\rho\rho}$

5) $$\Gamma'^{\lambda}_{\mu\nu} = \frac{\partial x'^{\lambda}}{\partial x^{\alpha}} \frac{\partial x^{\omega}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \Gamma^{\alpha}_{\omega\alpha} + \frac{\partial x'^{\lambda}}{\partial x^{\alpha}} \frac{\partial^2 x^{\alpha}}{\partial x'^{\mu} \partial x'^{\nu}}$$

Lista 1.

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1. $\phi(x)$ escalar: $\phi'(x') = \phi(x)$

$$(\vec{\nabla}\phi\, \vec{\nabla}\phi) \longrightarrow \vec{\nabla}'\phi' \cdot \vec{\nabla}'\phi' = \sum_i \frac{\partial}{\partial x'_i}\phi \frac{\partial}{\partial x'_i}\phi$$
$$= \left(R^{ki}\frac{\partial}{\partial x^i}\phi\right)\left(R^{kj}\frac{\partial\phi}{\partial x_j}\right)$$
$$= \underbrace{(R^{ik})^T R^{kj}}_{\delta^{ij}} \frac{\partial\phi}{\partial x^i}\frac{\partial\phi}{\partial x_j} = \frac{\partial\phi}{\partial x^i}\frac{\partial\phi}{\partial x^i} = \vec{\nabla}\phi \cdot \vec{\nabla}\phi$$

$$\nabla^2\phi' = \frac{\partial}{\partial x^i}\frac{\partial}{\partial x_j}\phi \longrightarrow R^{ki}R^{kj}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x_j}\phi$$
$$= \delta^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x_j}\phi = \nabla^2\phi$$

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3. $$\begin{cases} x = \dfrac{W}{2\pi}\varphi \\ y = -\dfrac{W}{2\pi}\log \operatorname{tg}\dfrac{\theta}{2} \end{cases}$$

$$(\theta, \varphi): \begin{pmatrix} 1 & \\ 0 & \sin^2\theta \end{pmatrix}$$
$$\rightarrow (x, y): g'_{\rho\sigma}(x) = $$
$$= g_{\mu\nu}(x) \frac{\partial x^{\mu}}{\partial x'^{\rho}} \frac{\partial x^{\nu}}{\partial x'^{\sigma}}$$

$$g'_{xx} = \left(\cancel{\frac{\partial\theta}{\partial x}}\right)^2 + \sin^2\theta \left(\frac{\partial\varphi}{\partial x}\right)^2$$

$$g'_{xy} = \left(\frac{\partial\theta}{\partial x}\right)\left(\frac{\partial\theta}{\partial y}\right) + \sin^2\theta \left(\frac{\partial\varphi}{\partial x}\right)\left(\frac{\partial\varphi}{\partial y}\right) = 0$$

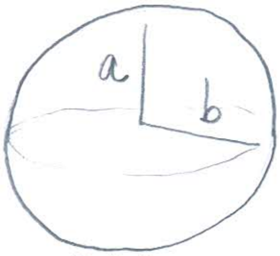
$$g'_{yy} = \left(\frac{\partial\theta}{\partial y}\right)^2 + \sin^2\theta \left(\cancel{\frac{\partial\varphi}{\partial y}}\right)^2 \quad 0$$

$$\Rightarrow g'_{\mu\nu} = \begin{pmatrix} \sin^2\theta \left(\frac{\partial\varphi}{\partial x}\right)^2 & 0 \\ 0 & \left(\frac{\partial\theta}{\partial y}\right)^2 \end{pmatrix}$$

$$\varphi = \frac{2\pi}{W} x \qquad \theta = 2\,tg^{-1}\, e^{-\frac{2\pi y}{W}}$$

$$= \Omega^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \Omega^2 = \frac{4\pi^2}{W^2} \frac{1}{Ch\left(\frac{2\pi y}{W}\right)}$$

12.



$a \neq b$

$$ds^2 = (b^2 + a^2\cos^2\theta)\, d\theta^2 + \frac{(b^2+a^2)^2}{b^2 + a^2\sin^2\theta} \sin^2\theta\, d\varphi^2$$

Equador: $\theta = \frac{\pi}{2}$

$d\theta = 0$

$$\longrightarrow ds^2 = \frac{(b^2+a^2)^2}{b^2} \cdot 1\, d\varphi^2$$

$$\int_0^{2\pi} d\varphi \frac{b^2+a^2}{b} = 2\pi \frac{b^2+a^2}{b}$$

Longitude: $\varphi = \text{const}$

$$d\varphi = 0$$

$$ds^2 = (b^2 + a^2\cos^2\theta)\,d\theta^2$$

integral elíptica completa

$$= 2\int_0^{\pi} d\theta \sqrt{b^2 + a^2\cos^2\theta} = 2\sqrt{a^2+b^2}\, E\left(\theta, \frac{a^2}{a^2+b^2}\right)\Bigg|_{\theta=0}^{\theta=\pi}$$

→ 2 arcos iguais

$$a = b \longrightarrow 2\sqrt{2}\, a\, E\left(\theta, \tfrac{1}{2}\right)\Big|_0^{\pi} = 2\sqrt{2}\,a\Big(\underbrace{E\left(\tfrac{1}{2}\right)}_{2{,}70129\ldots} - 0\Big)$$

$$= 2\pi a \Rightarrow E\left(\tfrac{1}{2}\right) = \frac{\pi}{\sqrt{2}}$$

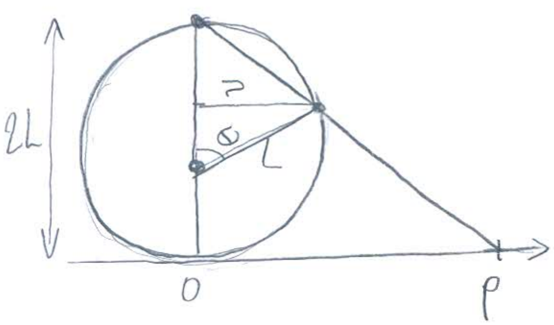
Area

$$\int_0^{\pi} d\theta \int_0^{2\pi} d\varphi \sqrt{(b^2+a^2\cos^2\theta)\frac{(b^2+a^2)^2\sin^2\theta}{(b^2+a^2\cos^2\theta)}} = 4\pi(b^2+a^2)$$

13) $$ds^2 = \frac{dr^2}{1 - r^2/L^2} + r^2 d\varphi^2$$

$$\left(\begin{array}{l} L\sin\theta = r \\ L\cos\theta\, d\theta = dr \end{array}\right. \rightarrow ds^2 = \frac{L^2\cos^2\theta\, d\theta^2}{1-\sin^2\theta} + L^2\sin^2\theta\, d\varphi^2$$

$$= L^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$



Similaridade de triângulo

$$\left\{\begin{array}{l} \dfrac{L(1-\cos\theta)}{r} = \dfrac{2L}{\rho} \\ \sin\theta = \dfrac{r}{L} \end{array}\right.$$

$$r = \frac{\rho}{1 + \rho^2/4L^2}$$

13) $$ds^2 = \frac{dr^2}{1-r^2/L^2} + r^2 d\varphi^2$$

$$dr = \frac{d\rho}{1+\rho^2/4L^2} - \frac{\rho}{(1+\rho^2/4L^2)^2}\frac{2\rho\, d\rho}{4L^2}$$

$$\dots = \frac{1-\rho^2/4L^2}{(1+\rho^2/4L^2)^2}\, d\rho$$

$$1 - r^2/L^2 = \frac{(1-\rho^2/4L^2)^2}{(1+\rho^2/4L^2)^2}$$

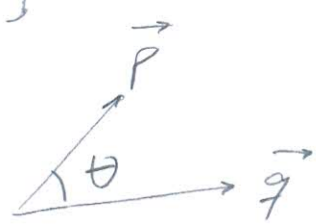
$$= \frac{1}{(1+\rho^2/4L^2)^2}\left(d\rho^2 + \rho^2 d\varphi^2\right)$$

14) $$g_{\mu\nu} = \Omega^2 \delta_{\mu\nu}$$

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = \Omega^2 d\vec{x}^{\,2}$$

$$ds^2_{flat} = \delta_{\mu\nu}\, dx^\mu dx^\nu = d\vec{x}^{\,2}$$

Distâncias são diferentes



Ângulos: $\vec{p}, \vec{q}$

$$\vec{p}\cdot\vec{q} = |\vec{p}||\vec{q}|\cos\theta$$

$$\cos\theta = \frac{\vec{p}\cdot\vec{q}}{|\vec{p}||\vec{q}|} = \frac{g_{\mu\nu}\, p^\mu q^\nu}{\sqrt{g_{\rho\sigma}\, p^\rho p^\sigma}\sqrt{g_{\lambda\tau}\, q^\lambda q^\tau}}$$

$$= \frac{\Omega^2 \vec{p}\cdot\vec{q}}{\sqrt{\Omega^2 \vec{p}^{\,2}}\sqrt{\Omega^2 \vec{q}^{\,2}}} = \vec{p}\cdot\vec{q}$$

Superfícies em Minkowski

$$F(x^\mu) = 0$$

→ tipo-espaço: a distância infinitesimal entre 2 pontos da superfície é tipo-espaço.

↳ $\begin{cases} \text{tangentes} \rightarrow \text{tipo espaço} \\ \text{normal} \rightarrow \text{tipo tempo} \end{cases}$

→ nulas (tipo-luz): geradas por vetores de tipo-luz

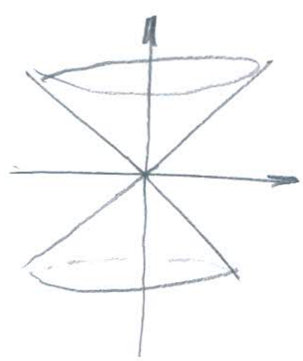
↳ { Normal é também de tipo luz e faz parte da Superfície

Ex: Minkowski em coord. esféricas

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2$$

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

Cone de Luz: $\left.\begin{array}{l} t = r \\ dt = dr \\ d\theta = 0 \\ d\varphi = 0 \end{array}\right\} \rightarrow ds^2 = 0$



trajetória: $l^\mu = (1,1,0,0)$

$l^\mu l^\nu g_{\mu\nu} = 0 \rightarrow$ vetor tangente ao cone de luz

Outros 2 vetores tangentes são:

$h^\mu = (0,0,\frac{1}{r},0)$ & $K^\mu = (0,0,0,\frac{1}{r\sin\theta})$

$h^\mu h^\nu g_{\mu\nu} = 1 = K^\mu K^\nu g_{\mu\nu} \rightarrow$ tipo-espaço; ortogonais a l^μ

$h^\mu l^\nu g_{\mu\nu} = 0 = K^\mu l^\nu g_{\mu\nu}$

Vetor normal: l^μ mesmo