Relatividade - 11/04/16

Transformações de Lorentz

- lineares
- coef. constantes

$$\begin{cases} p^\mu \to p'^\mu = \Lambda^\mu{}_\nu p^\nu \\ q^\mu \to q'^\mu = \Lambda^\mu{}_\nu q^\nu \end{cases}$$

$p \cdot q$ deve ser invariante

$$p \cdot q \to p' \cdot q' = \eta_{\mu\nu} p'^\mu q'^\nu = \eta_{\mu\nu} \Lambda^\mu{}_\sigma p^\sigma \Lambda^\nu{}_\rho q^\rho$$

$$\overset{!}{=} p \cdot q = \eta_{\sigma\rho} p^\sigma q^\rho$$

$\boxed{\eta_{\mu\nu} \Lambda^\mu{}_\sigma \Lambda^\nu{}_\rho = \eta_{\sigma\rho}}$ Definição de T. L.

$$\Lambda^\mu{}_\sigma \eta_{\mu\nu} \Lambda^\nu{}_\rho = \mu_{\sigma\rho}$$

$$(\Lambda_\sigma{}^\mu)^T \eta_{\mu\nu} \Lambda^\nu{}_\rho = \eta_{\sigma\rho}$$

$\boxed{\Lambda^T \eta \Lambda = \eta}$ Forma matricial

Similar a: $R^T \mathbb{1} R = \mathbb{1}$ $[SO(D)]$

Mais em geral:

$$g'(x') = (S^{-1})^T g(x) S^{-1}$$

R, Λ são as S^{-1} que preservam: $\mathbb{1}, \eta$.

Algebra $\rightarrow$ transformações infinitesimais & geradores

$$\Lambda^{\mu}{}_{\sigma} \cong \delta^{\mu}{}_{\sigma} + \varphi K^{\mu}{}_{\sigma} + \mathcal{O}(\varphi^2)$$

parâmetro gerador

$$\eta_{\rho\mu}\eta^{\mu\nu} = \delta^{\nu}{}_{\rho}$$

(Lembre: $R \simeq \mathbb{1} + \theta J + \mathcal{O}(\theta^2)$)

$$\eta_{\mu\nu}\Lambda^{\mu}{}_{\sigma}\Lambda^{\nu}{}_{\rho} = \eta_{\sigma\rho}$$

$$\eta_{\mu\nu}(\delta^{\mu}{}_{\sigma} + \varphi K^{\mu}{}_{\sigma})(\delta^{\nu}{}_{\rho} + \varphi K^{\nu}{}_{\rho}) = \eta_{\sigma\rho}$$

$$K^{\mu}{}_{\sigma}\eta_{\mu\rho} + \eta_{\sigma\nu}K^{\nu}{}_{\rho} = 0$$

$$\boxed{K^T \eta = -\eta K}$$

$$K_{\mu\nu} \begin{pmatrix} K_{00} & K_{01} & K_{02} & K_{03} \\ K_{10} & K_{11} & K_{12} & K_{13} \\ K_{20} & K_{21} & K_{22} & K_{23} \\ K_{30} & K_{31} & K_{32} & K_{33} \end{pmatrix} \qquad \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\mu, \nu = \underbrace{0}_{\text{tempo}}, \underbrace{1, 2, 3}_{\text{espaço}}$

2 Direções espaciais

$$\begin{pmatrix} K_{22} & K_{32} \\ K_{23} & K_{33} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} K_{22} & K_{23} \\ K_{32} & K_{33} \end{pmatrix}$$

antissimétrico

$$K_{22} = -K_{22} \quad\rightarrow\quad K_{22} = 0 \qquad K_{32} = -K_{23}$$

$$K_{33} = -K_{33} \qquad\quad K_{33} = 0$$

$$\Rightarrow J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Rotações!

1 tempo + 1 espaço

$$\begin{pmatrix} K_{00} & K_{10} \\ K_{01} & K_{11} \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} K_{00} & K_{01} \\ K_{10} & K_{11} \end{pmatrix}$$

$$\begin{pmatrix} -K_{00} & K_{10} \\ -K_{01} & K_{11} \end{pmatrix} = -\begin{pmatrix} -K_{00} & -K_{01} \\ K_{10} & K_{11} \end{pmatrix} \Rightarrow \boxed{K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$$

Boost

Simétrica

Transformações - finitas

$$\varphi = \frac{\Phi}{N}$$ ← finito, ← grande

Infinitesimal

$$\begin{pmatrix} K_{00} & K_{01} \\ K_{10} & K_{11} \end{pmatrix}$$

$$\Lambda(\varphi) \simeq 1 + \varphi K = 1 + \frac{\Phi}{N} K$$

$$\Lambda(\Phi) = \left[\Lambda(\varphi)\right]^N \xrightarrow[N\to\infty]{} e^{\Phi K}$$

$$= \sum_{m=0}^{\infty} \frac{\Phi^m K^m}{m!} = \left(\sum_{K=0}^{\infty} \frac{\Phi^{2K}}{(2K)!}\right) + K\left(\sum_{K=0}^{\infty} \frac{\Phi^{2K+1}}{(2K+1)!}\right)$$

fixado (0,1)

$$= \operatorname{ch}\Phi + K \operatorname{sh}\Phi = \begin{pmatrix} \operatorname{ch}\Phi & \operatorname{sh}\Phi \\ \operatorname{sh}\Phi & \operatorname{ch}\Phi \end{pmatrix}$$

Lorentz é um grupo:

1) Composição
$$\begin{cases} x^\mu \to \Lambda^\mu{}_\nu x^\mu = x'^\mu \\ x'^\mu \to x''^\mu = \bar\Lambda^\mu{}_\nu x'^\nu \end{cases}$$

$$x''^\mu = \underbrace{\bar\Lambda^\mu{}_\nu \Lambda^\nu{}_\rho}_{\bar{\bar\Lambda}^\mu{}_\rho \text{ que é uma t.L}} x^\rho$$

2) $(\det \Lambda)^2 = 1$

$\rightarrow \det \Lambda = \pm 1 \neq 0$

$\exists$ inversa $\quad (\Lambda^{-1})^{\rho}{}_{\nu} = \Lambda_{\nu}{}^{\rho}$

3) $\exists$ identidade $\underline{\Lambda = 1}$

Escolhemos:

•) $\boxed{\det \Lambda = +1}$ → Transformação de Lorentz Própria

•) $\eta_{\mu\nu} \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} = \eta_{\rho\sigma}$

$(\Lambda^0{}_0)^2 = 1 + \Lambda^i{}_0 \Lambda^i{}_0 \longrightarrow \Lambda^0{}_0 = \pm 1$

$\boxed{\Lambda^0{}_0 = +1}$ T.L. Ortocrôna

SO(D): Deixa invariante $d\vec{x}^{\,2} = \sum_{i=1}^{D} (dx_i)^2$

$ds^2 = -\sum_{i=1}^{m} (dx_i)^2 + \sum_{i=m+1}^{n} (dx_i)^2$

↳ $SO(m,n)$

Lorentz: $\boxed{SO(1,3)}$ (ou $SO(3,1)$ convenção)

$$\Lambda = 1 + i\theta^{\mu\nu} J_{\mu\nu} + \mathcal{O}(\theta^2) \longrightarrow \text{geral}$$

Algebra: $SO(1, D)$

$$\underbrace{[J_{\mu\nu}, J_{\rho\sigma}]}_{J_{\mu\nu}J_{\rho\sigma} - J_{\rho\sigma}J_{\mu\nu}} = i\left(\eta_{\mu\rho} J_{\nu\sigma} + \eta_{\nu\sigma} J_{\mu\rho} - \eta_{\nu\rho} J_{\mu\sigma} - \eta_{\mu\sigma} J_{\nu\rho}\right)$$

("a algebra fecha")
a diferença de duas operações é operação na algebra!

$D = 3$ $\quad SO(1, 3)$

1t+1e: $J_{0i} \equiv K_i$ ③ boosts $i = (1, 2, 3)$ ou (x, y, z)

2e: $J_{jk} \equiv \frac{1}{2}\epsilon_{ijk} J_{jk} \equiv J^i$ ③ rotações

$\Rightarrow$ o grupo de Lorentz tem 6 geradores

Por exemplo:

$$[J_x, J_y] = [J_{23}, J_{31}] = -i\eta_{33} J_{21} = iJ_{12}$$
$$= iJ_z$$

$$[J_z, K_x] = [J_{12}, J_{01}] = -i\eta_{11} J_{20} = iK_y$$

$$[K_x, K_y] = [J_{01}, J_{02}] = i\eta_{00} J_{12} = -iJ_z$$

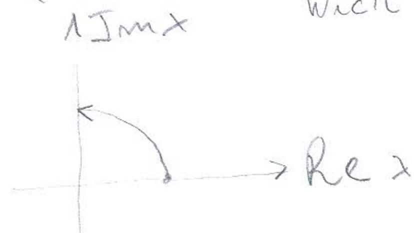
+ 4 translações $\quad P_\mu \begin{cases} P_0 = H \\ P_i \text{ momento linear} \end{cases}$

$\boxed{\begin{array}{l}\text{3 boost +}\\ \text{3 rotações +}\\ \text{4 translações}\end{array}}$

<u>Nota</u>: $R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ $\quad K = \begin{pmatrix} Ch\varphi & Sh\varphi \\ Sh\varphi & Ch\varphi \end{pmatrix}$

Rotação de Wick $\quad \underset{\text{Espaço-tempo}}{(t = x^0, x^1)} \xrightarrow[\text{wick}]{} \underset{\text{Espaço puro}}{(x^2, x^1)}$

$$\begin{cases} t = x^0 = i x^2 \\ \varphi = i\theta \end{cases}$$



$$Ch\varphi = \frac{e^{\varphi} + e^{-\varphi}}{2} \longrightarrow \frac{e^{i\theta} + e^{-i\theta}}{2} = \cos\theta$$

$$Sh\varphi = \frac{e^{\varphi} - e^{-\varphi}}{2} \longrightarrow \frac{e^{i\theta} - e^{-i\theta}}{2} = i\sin\theta$$

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} Ch\varphi & Sh\varphi \\ Sh\varphi & Ch\varphi \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}$$

$$\begin{pmatrix} i x_2' \\ x_1' \end{pmatrix} = \begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} i x_2 \\ x_1 \end{pmatrix}$$

$$\begin{pmatrix} x_2' \\ x_1' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$$

$$SO(1,1) \longrightarrow SO(2)$$

$ds^2 = -dt^2 + d\vec{x}^2$

$t \to i y$

$= dy^2 + d\vec{x}^2$

Lorentz

$\downarrow$

Euclid.

Ação para uma partícula não-relativística

$$S_{NR} = \int dt\, L = \int dt \left(\frac{m}{2} \left(\frac{d\vec{x}}{dt} \right)^2 - V(\vec{x}) \right)$$

Por enquanto: $V(\vec{x}) = 0$

Generalização relativística?

$$\sqrt{a^2 - \epsilon^2} \underset{\epsilon \ll a}{\simeq} a - \frac{\epsilon^2}{2a} + \dots \Rightarrow \frac{\epsilon^2}{2a} = a - \sqrt{a^2 - \epsilon^2}$$

Por motivos dimensionais → precisamos introduzir $\underline{\underline{c}}$.

$$\frac{(d\vec{x})^2}{2dt} = c\, \frac{(d\vec{x})^2}{2c\,dt} \qquad \begin{matrix} \epsilon \longleftrightarrow dx \\ a \longleftrightarrow c\,dt \end{matrix}$$

Não relativ. $\epsilon \ll a$

$$= c \left(c\,dt - \underbrace{\sqrt{c^2 dt^2 - d\vec{x}^2}}_{\text{tempo próprio!}} + \dots \right)$$

Proposta: $\boxed{S = -mc \int \sqrt{c^2 dt^2 - d\vec{x}^2} + \underset{0}{mc^2 \int dt}}$

$\delta t_i = \delta t_f = 0$

$$\boxed{S_{rel.} = -mc^2 \int dt \sqrt{1 - \left(\frac{d\vec{x}}{c\,dt} \right)^2}}$$

inv. Lorentz ⊕
inv. reparametrização.

Teste: limite N.R.

$$S_{rel} \simeq -mc^2 \int dt \left(1 - \frac{1}{2} \left(\frac{d\vec{x}}{c\,dt} \right)^2 + \dots \right)$$

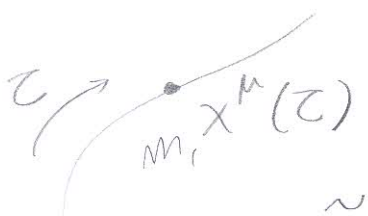
$$= \underbrace{-mc^2}_{\text{"alguma" energia}} \int dt + \frac{m}{2} \int dt \left(\frac{d\vec{x}}{dt} \right)^2 + \dots \quad ✓$$

Que tal $m=0$? $(c=1)$

Sem dinâmica

$S_{rel} = 0$

→ Vamos introduzir um objeto auxiliar $e(\tau)$ métrica ao longo da linha de mundo



$$\tilde{S}_{rel} = \frac{1}{2}\int d\tau\left(e^{-1}\dot{x}^2 - m^2 e\right)$$

↳ $\dot{x}^2 = \eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$

[e não tem 2 derivadas temporais (sem dinâmica)]

eqs. do movimento de $e(\tau)$: (algébrica, não diferencial)

$$\frac{d}{dt}\left(\frac{\delta L}{\delta \dot{e}}\right) = \frac{\delta L}{\delta e} \longrightarrow m^2 e^2 + \dot{x}^2 = 0$$

$$e^2 = -\frac{\dot{x}^2}{m^2}$$

$$\tilde{S}_{rel} = \frac{1}{2}\int d\tau\left(e^{-1}\dot{x}^2 - m^2 e\right)$$

$$= \frac{1}{2}\int d\tau\left(\sqrt{-\frac{m^2}{\dot{x}^2}}\,\dot{x}^2 - m^2\sqrt{-\frac{\dot{x}^2}{m^2}}\right)$$

$$= -m\int d\tau\sqrt{-\dot{x}^2} = -m\int d\tau\sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}$$

$$= -m\int\sqrt{-\eta_{\mu\nu}dx^\mu dx^\nu} = -m\int\sqrt{dt - d\vec{x}^2} = S_{rel}$$

S_{rel} e $\tilde{S}_{rel}$ são equivalentes na "camada de massa" de e

↳ Usando as eqs do mov. de e (on-shell) ?