Relatividade - 01/05

Tensores sobre T.G.C.

T.G.C. $dx'^{\mu} = \dfrac{\partial x'^{\mu}}{\partial x^{\nu}} dx^{\nu} \equiv S^{\mu}{}_{\nu}(x)\, dx^{\nu}$

Vetor sob. T.G.C. $W'^{\mu}(x') = S^{\mu}{}_{\nu}(x)\, W^{\nu}(x)$

mesmo ponto mas coordenadas x^{μ} & x'^{μ}

Tensor sob T.G.C.: $T'^{\mu\nu}(x') = S^{\mu}{}_{\rho}(x)\, S^{\nu}{}_{\sigma}(x)\, T^{\rho\sigma}(x)$

índices contravariantes (= up)

Exemplo de tensor com índices covariantes:

↳ métrica

$$g'_{\rho\sigma}(x') = g_{\mu\nu}(x)\, (S^{-1})^{\mu}{}_{\rho}(x)\, (S^{-1})^{\nu}{}_{\sigma}(x)$$

índice contrav. $\rightarrow S$

índice covariante $\rightarrow S^{-1}$

$$W_{\mu} = g_{\mu\nu} W^{\nu}$$

↳ vetor covariante associado ao vetor contravariante W^{ν}

$$\underline{\text{Teste}}: \quad W'_\rho = g'_{\rho\sigma} W'^\sigma = \underbrace{g_{\mu\nu}(S^{-1})^\mu{}_\rho (S^{-1})^\nu{}_\sigma}_{g'_{\rho\sigma}} \underbrace{S^\sigma{}_\lambda W^\lambda}_{W'^\sigma}$$

$$= W_\mu (S^{-1})^\mu{}_\rho \quad \underline{\underline{\text{OK}}}$$

*) $W_\mu V^\mu$ é um escalar

$$\parallel$$

$$g_{\mu\nu} W^\mu V^\nu = W^\mu V_\mu = g^{\mu\nu} W_\mu V_\nu$$

$$\underline{\text{Teste}}. \quad W'_\rho V'^\rho = W_\mu \underbrace{(S^{-1})^\mu{}_\rho S^\rho{}_\nu}_{\delta^\mu{}_\nu} V^\nu = W_\nu V^\nu$$

*) $\partial_\mu \phi(x)$ é um vetor

$\boxed{\phi'(x') = \phi(x)}$ escalar

$$\partial'_\mu \phi'(x') = \frac{\partial \phi'(x')}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial \phi(x)}{\partial x^\nu} = (S^{-1})^\nu{}_\mu \partial_\nu \phi(x)$$

$$\begin{cases} dx^\mu \to \text{vetor contrav.} \\ \partial_\mu = \dfrac{\partial}{\partial x^\mu} \to \text{vetor cov.} \end{cases}$$

*) $\partial_\lambda W^\mu$ é um tensor? $\underline{\text{Não}}$

$$\partial'_\lambda W'^\mu = \underbrace{\frac{\partial x^\nu}{\partial x'^\lambda} \frac{\partial}{\partial x^\nu}}_{\partial'_\lambda} \underbrace{\left(S^\mu{}_\rho W^\rho\right)}_{W'^\mu} = (S^{-1})^\nu{}_\lambda \frac{\partial}{\partial x^\nu}\left(S^\mu{}_\rho(x) W^\rho\right)$$

extra! → S depende de x

$$= (S^{-1})^\nu{}_\lambda S^\mu{}_\rho(x) \partial_\nu W^\rho + \overbrace{(S^{-1})^\nu{}_\lambda \left(\frac{\partial}{\partial x^\nu} S^\mu{}_\rho\right) W^\rho}^{\text{extra!}}$$

→ Derivada Covariante

D_μ : $D_\lambda W^\mu$ é um tensor!

$$D_\lambda W^\mu \longrightarrow \boxed{D'_\lambda W'^\mu = (S^{-1})^\nu{}_\lambda \, S^\mu{}_\rho \, D_\nu W^\rho}$$

$D_\mu = ?$ $\quad D_\mu = \partial_\mu + \ldots$

para cancelar o pedaço extra

Ansatz

Linear em W^μ:

constante

Não pode ser um tensor

$$D_\lambda W^\mu = \partial_\lambda W^\mu + \underbrace{\# \, \tilde{\Gamma}^\mu_{\lambda\nu}}_{} W^\nu$$

Símbolos de Christoffel

$$\boxed{D_\lambda W^\mu = \partial_\lambda W^\mu + \Gamma^\mu_{\lambda\nu} W^\nu}$$

$$\boxed{D_\lambda W_\mu = \partial_\lambda W_\mu - \Gamma^\sigma_{\mu\lambda} W_\sigma}$$

(HW) Check!

Use que $MM^{-1} = \mathbb{1}$

$$\partial(MM^{-1}) = \partial(\mathbb{1}) = 0$$

$$(\partial M)M^{-1} + M\,\partial M^{-1} = 0$$

$$\boxed{\partial M^{-1} = -M^{-1}(\partial M)M^{-1}}$$

*) $\boxed{D_\lambda g_{\mu\nu} = 0}$ (HW)

Definição de Γ como "metric connection"

$$\partial_\lambda g_{\mu\nu} - \Gamma^\sigma_{\lambda\mu} g_{\sigma} - \Gamma^\sigma_{\lambda\nu} g_{\sigma}$$

$$D_\lambda T_{\mu\nu}{}^\rho = \partial_\lambda T_{\mu\nu}{}^\rho - \underbrace{\Gamma T}_{\mu} - \underbrace{\Gamma T}_{\nu} + \underbrace{\Gamma T}_{\rho}$$

(HW) Arrumar os índices

*) Maxwell no espaço-tempo curvo

Plano: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

Promoção de plano para curvo.
$\partial_\mu \longrightarrow D_\mu$

Curvo: $F_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu$

$$= \partial_\mu A_\nu - \cancel{\Gamma^\sigma_{\mu\nu} A_\sigma} - (\partial_\nu A_\mu - \cancel{\Gamma^\sigma_{\nu\mu} A_\sigma})$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu$$ Devido à antissimetria de $F_{\mu\nu}$ e à simetria de $\Gamma_{\mu\nu}$.

Plano: $-\frac{1}{4}\int d^4x\, F_{\mu\nu}F^{\mu\nu}$

Curvo: $$\boxed{-\frac{1}{4}\int d^4x \sqrt{-\det g}\; g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma}}$$

$= S_{maxwell}$ (parte do campo)

*) Div. covariante:

$$D_\mu W^\mu = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\, W^\mu\right)$$ (Ver aula anterior)

mas também:

$$D_\mu W^\mu = \partial_\mu W^\mu + \Gamma^\mu_{\mu\nu} W^\nu$$

$$\rightarrow \boxed{g^{\mu\sigma}\partial_\nu g_{\mu\sigma} = \frac{1}{2}\partial_\nu g}$$ (HW)

↳ Precisa usar: $\log \det M = \operatorname{tr} \log M$ (HW)

Equações de Einstein

- → Ação de Einstein-Hilbert
- → Variação → equações

Até agora: $S_{Newton} = \int dt\, d^3\vec{x}\, (\vec{\nabla}\phi)^2 \rightarrow \phi$

$S_{Maxwell} = \int d^4x\, F_{\mu\nu}F^{\mu\nu} \rightarrow A_\mu$

1) Quadráticas nos campos

2) Quadráticas nas derivadas ✓

$\Downarrow$

Encontrar uma ação com 2 derivadas da métrica e $(\vec{\nabla}\phi)^2$ como limite não-relativístico.

$\phi, A_\mu \longrightarrow 0$ na ausência dos campos

$g_{\mu\nu} \rightarrow \eta_{\mu\nu}$

$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ → papel análogo a ϕ e A_μ

$g^{\mu\nu}$ vai aparecer $\longrightarrow$ $\eta^{\mu\nu} - h^{\mu\nu}$ vai aparecer

$\downarrow$

$(h_{\mu\nu} + h_{\mu\nu})^{-1}$ → Série infinita em $h_{\mu\nu}$

→ expectativa: a ação não vai ser quadrática em $h_{\mu\nu}$, mas mais complicada.

*) T.G.C.: a ação deve ser invariante sob T.G.C.

ação ↔ Curvatura
2 derivadas da métrica
mais complicada que "g^2"
Invariante sob. T.G.C.

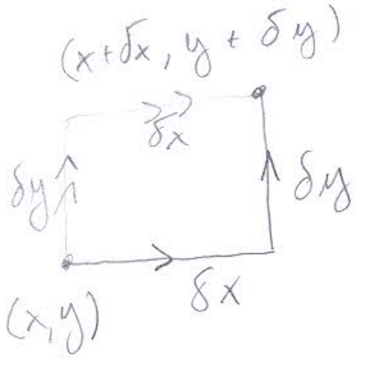
Aliás, não podemos usar $D_\mu g_{\lambda\sigma} = 0$
→ Vai ter combinações de $\partial_\mu g_{\lambda\sigma}$

$\boxed{\text{Curvatura}}$

Exemplos

1D $$f(x) \longrightarrow f(x+\delta x) = f(x) + \delta x \frac{df}{dx}$$
$$= \underbrace{\left(1 + \delta x \frac{d}{dx}\right)}_{\text{operador de Translação}} f(x)$$

2D $$f(x,y) \longrightarrow f(x+\delta x, y+\delta y)$$



$$= \underbrace{\left(1 + \delta y \frac{\partial}{\partial y}\right)}_{\text{Translação em y depois.}} \underbrace{\left(1 + \delta x \frac{\partial}{\partial x}\right)}_{\text{Translação em x primeiro}} f(x,y)$$

$$= \left(1 + \delta y \frac{\partial}{\partial y} + \delta x \frac{\partial}{\partial x} + \delta y\, \delta x \frac{\partial}{\partial y}\frac{\partial}{\partial x}\right) f(x,y)$$

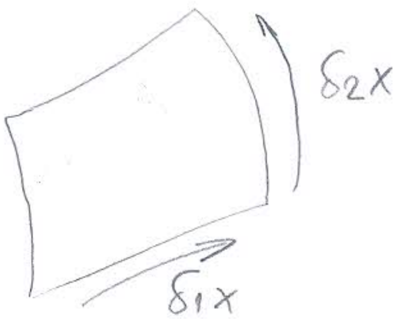
Transl. em y primeiro e depois x
$$(x \longleftrightarrow y)\, f(x,y)$$

$$(x \text{ depois } y)\, f(x,y) - (y \text{ depois } x)\, f(x,y) = 0$$

$$\delta x\, \delta y \underbrace{\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right]}_{=0} f(x,y) = 0$$

óbvio!

Plano $\rightarrow$ Curvo



Transladar um vetor $S_\rho(x)$

$$(1+\delta_2 x^\nu D_\nu)(1+\delta_1 x^\mu D_\mu) - (1+\delta_1 x^\mu D_\mu)(1+\delta_2 x^\nu D_\nu)\, S_\rho(x)$$

$$= \delta_2 x^\mu \delta_1 x^\nu \underbrace{[D_\mu , D_\nu]}_{\neq 0} S_\rho(x)$$

$$\boxed{[D_\mu , D_\nu]\, S_\rho = (D_\mu D_\nu - D_\nu D_\mu)\, S_\rho}$$

$$D_\mu D_\nu S_\rho = \partial_\mu (D_\nu S_\rho) - \Gamma^\sigma_{\mu\nu} D_\sigma \cancel{S_\rho} - \Gamma^\sigma_{\mu\rho} D_\nu S_\sigma$$

$$-D_\nu D_\mu S_\rho = -\partial_\nu (D_\mu S_\rho) + \Gamma^\sigma_{\nu\mu} \cancel{D_\sigma} S_\rho + \Gamma^\sigma_{\nu\rho} D_\mu S_\sigma$$

$$\rightarrow [D_\mu , D_\nu]\, S_\rho = \partial_\mu (\cancel{\partial_\nu S_\rho} - \Gamma^\sigma_{\nu\rho} S_\sigma) - \Gamma^\sigma_{\mu\rho}(\partial_\nu S_\sigma - \Gamma^\lambda_{\nu\sigma} S_\lambda) + - (\mu \leftrightarrow \nu)$$

$$= -(\partial_\mu \Gamma^\sigma_{\nu\rho}) S_\sigma \overset{\mu\leftrightarrow\nu}{- \Gamma^\sigma_{\nu\rho} \cancel{\partial_\mu} S_\sigma - \Gamma^\sigma_{\mu\rho} \cancel{\partial_\nu} S_\sigma} + \Gamma^\sigma_{\mu\rho} \Gamma^\lambda_{\nu\sigma} S_\lambda - (\mu - \nu)$$

$$= -\left(\partial_\mu \Gamma^\sigma_{\nu\rho} - \Gamma^\lambda_{\mu\rho} - (\mu \leftrightarrow \nu)\right) S_\sigma \equiv - R^\sigma{}_{\rho\mu\nu} S_\sigma$$

$$\boxed{R^\sigma{}_{\rho\mu\nu} = \partial_\mu \Gamma^\sigma_{\nu\rho} + \Gamma^\sigma_{\mu\lambda}\Gamma^\lambda_{\nu\rho} - (\mu \leftrightarrow \nu)} \sim \text{"}\partial g \partial g + \partial^2 g\text{"}$$

1) 2 derivadas da métrica

2) Curvatura do espaço

3) É um tensor (Definido em termos de "coisas" covariantes) $[D_\mu, D_\nu] S_\rho$

4) É único: imagine que $\exists\ \tilde{R}^\sigma{}_{\rho\mu\nu}$ que é um tensor de 2 derivadas da métrica

$$\Delta^\sigma{}_{\rho\mu\nu} \equiv R^\sigma{}_{\rho\mu\nu} - \tilde{R}^\sigma{}_{\rho\mu\nu} \xrightarrow[\text{ref. localmente plano}]{} \boxed{\Delta^\sigma{}_{\rho\mu\nu} = 0}$$

↳ é válido em qualquer referencial

Propriedades

1) $R^\lambda{}_{\rho\mu\nu} = -R^\lambda{}_{\rho\nu\mu}$ por definição

↳ $\boxed{R_{\tau\rho\mu\nu} = g_{\tau\lambda} R^\lambda{}_{\rho\mu\nu}}$ (A) $\longrightarrow \underset{\tau}{4} \times \underset{\rho}{4} \times \underset{\mu\nu}{\dfrac{(4\times 3)}{2}}$

$= 96$ Componentes

↳ coordenadas localmente planas

$x \approx 0$

$$g_{\tau\mu}(x) = \eta_{\tau\mu} + B_{\tau\mu\lambda\sigma} x^\lambda x^\sigma + \dots$$

$$\text{"}\begin{cases} \Gamma = 0 \\ \partial\Gamma \neq 0 \end{cases}\text{"} \Rightarrow R_{\tau\rho\mu\nu} = B_{\tau\nu\mu\rho} - B_{\rho\nu\mu\tau} - (\mu \leftrightarrow \nu)$$

$$\boxed{R_{\tau\rho\mu\nu} = -R_{\rho\tau\mu\nu}} \quad (B)$$

$$\boxed{R_{\tau\rho\mu\nu} + R_{\tau\mu\nu\rho} + R_{\tau\nu\rho\mu} = 0} \quad (D)$$

$$\boxed{R_{\tau\rho\mu\nu} = R_{\mu\nu\tau\rho}} \quad (C)$$

$$\overset{(A)}{96 \text{ componentes}} \xrightarrow{(B)} \left(\frac{4\times 3}{2}\right)\times\left(\frac{4\times 3}{2}\right) = 36 \text{ comp.} \xrightarrow{(C)} \underset{(HW)}{21} \xrightarrow{(D)} \underline{\underline{20}}$$

$$\boxed{R^{\sigma}{}_{\rho\mu\nu} = \partial_\mu \Gamma^{\sigma}_{\nu\rho} + \Gamma^{\sigma}_{\mu\lambda}\Gamma^{\lambda}_{\nu\rho} - (\mu \leftrightarrow \nu)}$$

<u>Tensor de Riemann</u>