Relatividade 04/05

$$\begin{cases} R^{\sigma}{}_{\rho\mu\nu} = (\partial_\mu \Gamma^{\sigma}_{\nu\rho} + \Gamma^{\sigma}_{\mu\lambda}\Gamma^{\lambda}_{\nu\rho}) - (\mu \leftrightarrow \nu) \\ R_{\tau\rho\mu\nu} = g_{\tau\lambda} R^{\lambda}{}_{\rho\mu\nu} \end{cases}$$

↳ Tensor de Riemann

"$R \sim (\partial g)^2 + \partial^2 g$"

Ação para a métrica

1) Escalar

$$\int d^4x \longrightarrow \int d^4x \sqrt{-g} \text{ invariante sob T.G.C.}$$

$g \equiv \det g_{\mu\nu}$

$-$ para $\sqrt{\ } \in \mathbb{R}$

$$S_{\text{gravidade}} = \int d^4x \sqrt{-g}\, A(x)$$

↳ Escalar

$$\begin{cases} A(x) = A'(x') \\ A(x) \text{ deve conter 2 derivadas da métrica} \end{cases}$$

$A(x)$ deve ser construído a partir de $R_{\mu\nu\rho\sigma}$

Quantos escalares podemos construir?

$$g^{\mu\nu} R_{\mu\nu\rho\sigma} = 0 = g^{\rho\sigma} R_{\mu\nu\rho\sigma}$$

$$\boxed{g^{\mu\rho} R_{\mu\nu\rho\sigma} \equiv R_{\nu\sigma}}$$ Tensor de Ricci $\overset{?}{=} R_{\sigma\nu}$ Simétrico

$$g^{\mu\sigma} R_{\mu\nu\rho\sigma} = - g^{\mu\sigma} R_{\mu\nu\sigma\rho} = - R_{\nu\rho}$$

$$\boxed{R \equiv g^{\nu\sigma} R_{\nu\sigma}}$$ Escalar de Curvatura (de Ricci)

$$\Rightarrow S_{\text{gravidade}} = \underset{?}{\#} \int d^4x \sqrt{-g}\, R$$

$\boxed{c = 1}$

$[S_{\text{gravidade}}] = M . L$ $\quad$ (ex. $S = -\underset{M}{m} \int \underset{L}{d\tau}$)

$[g] = 1$ $\quad$ $[S_{\text{grav.}}] = [\# \int d^4x \sqrt{-g}\, R]$

$[R] = L^{-2}$ $\quad$ $ML = [\#]\, L^4 . 1 . L^{-2}$.

$[\#] = \frac{ML}{L^2} = \frac{M}{L}$ $\quad$ $[G_N] = \frac{L}{M}$ $\quad$ $\# = \frac{a}{G_N}$ nº adimens.

a é determinado pelo limite não relativístico.

$$\boxed{\underset{\text{Einstein-Hilbert.}}{S_{EH}} = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g}\, R \quad (+ S_{\text{matéria}})}$$

é o equivalente de $\int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}\right)$

Parte de campo

"Navalha de Occam" (Occam's Razor)

Equações de Einstein

$$S_{EH} = \frac{1}{16\pi G}\int d^4x\, \underbrace{\sqrt{-g}}_{\mathrm{I}}\ \overbrace{\underbrace{g^{\mu\nu}}_{\mathrm{II}}\ \underbrace{R_{\mu\nu}}_{\mathrm{III}}}^{R}$$

$$\delta S_{EH} = \delta S_{EH}\big|_{\mathrm{I}} + \delta S_{EH}\big|_{\mathrm{II}} + \delta S_{EH}\big|_{\mathrm{III}}$$

II - $\dfrac{1}{16\pi G}\displaystyle\int d^4x\sqrt{-g}\, R_{\mu\nu}\, \delta g^{\mu\nu}$ $= -g^{\mu\rho}\delta g_{\rho\sigma}g^{\sigma\nu}$

$\to \delta(MM^{-1} = 1)$

$\delta M^{-1} = -M^{-1}\delta M M^{-1}$

$$\boxed{\delta S_{EH}\big|_{\mathrm{II}} = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}\, R^{\mu\nu}\,\delta g_{\mu\nu}}$$

I - $\delta(\sqrt{-g}) = \frac{1}{2}\sqrt{-g}\, g^{\mu\nu}\delta g_{\mu\nu}$

(HW) $\begin{cases} \log\det M = \mathrm{tr}\log M \\ \delta \det M = \det M\,\delta(\mathrm{tr}\log M) = \det M\,\mathrm{tr}(M^{-1}\delta M) \end{cases}$

$$\delta S_{EH}\big|_{\mathrm{I}} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}R\right)\delta g_{\mu\nu}$$

$$\delta S_{EH}\big|_{\mathrm{I}} + \delta S_{EH}\big|_{\mathrm{II}} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right]\delta g_{\mu\nu}$$

III) $\delta R_{\mu\nu} = R_{\mu\nu}(g+\delta g) - R_{\mu\nu}(g)$

$$\Gamma^{\rho}_{\nu\mu} \longrightarrow \Gamma^{\rho}_{\nu\mu} + \delta\Gamma^{\rho}_{\nu\mu} = \tilde{\Gamma}^{\rho}_{\nu\mu}$$

$\delta\Gamma^{\rho}_{\nu\mu} = \tilde{\Gamma}^{\rho}_{\nu\mu} - \Gamma^{\rho}_{\nu\mu}$ é um tensor (HW)

$$D_\lambda \delta \Gamma^\rho_{\nu\mu} = \partial_\lambda \delta \Gamma^\rho_{\nu\mu} + \Gamma^\rho_{\lambda\sigma} \delta \Gamma^\sigma_{\nu\mu} - \Gamma^\sigma_{\lambda\nu} \delta \Gamma^\rho_{\sigma\mu} - \Gamma^\sigma_{\lambda\mu} \delta \Gamma^\rho_{\nu\sigma}$$

$$(HW)\ \delta R^\rho{}_{\mu\lambda\nu} = D_\lambda (\delta \Gamma^\rho_{\nu\mu}) - D_\nu (\delta \Gamma^\rho_{\lambda\mu})$$

$$\delta S_{EH}\big|_{III} = \frac{1}{16\pi G} \int d^4x \sqrt{-g}\, g^{\mu\nu} \left(D_\lambda \delta \Gamma^\lambda_{\nu\mu} - D_\nu \delta \Gamma^\lambda_{\lambda\mu} \right)$$

$$\underset{\substack{\downarrow \\ (HW)}}{= \cdots} = \frac{1}{16\pi G} \int d^4x \sqrt{-g}\, D_\sigma \left(g_{\mu\nu} D^\sigma \delta g^{\mu\nu} - D_\lambda \delta g^{\sigma\lambda} \right)$$

$$= \frac{1}{16\pi G} \int d^4x \sqrt{-g}\, D_\sigma V^\sigma \doteq 0$$

↳ Stokes (menos termos de borda)

quando tem borda os termos de borda são importantes

↳ $\boxed{S_{GH} = \text{ação de Gibbons-Hawking}}$

$$\delta S_{EH}\big|_{III} = 0$$

$$\Rightarrow \frac{1}{16\pi G} \int d^4x \sqrt{-g} \underbrace{\left[R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right]}_{=0} \delta g_{\mu\nu}$$

$$\Rightarrow \boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0}$$

Equações de Einstein (gravidade pura, no vácuo sem matéria)

traco: $g^{\mu\nu} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = 0$

$$R - \frac{1}{2} \underbrace{g^{\mu\nu} g_{\mu\nu}}_{d} R = 0$$

d = número de dimensões do espaço-tempo

$d=4 \quad R - \frac{4}{2}R = 0 \Rightarrow \boxed{R=0}$ no vácuo

$\Rightarrow \boxed{R_{\mu\nu}=0}$

$R_{\mu\nu}=0 \not\Rightarrow R_{\mu\nu\rho\sigma}=0$

↳ espaço-tempo plano

$\underset{\underset{0}{\parallel}}{R_{\mu\nu}} = g^{\rho\sigma} R_{\rho\mu\sigma\nu}$ = Soma de termos positivos ou negativos

!

A soma pode ser zero sem os termos da soma serem zero.

Estrela esférica + estática

Fora da estrela: $\to$ vácuo

$$ds^2 = -A(r)\,dt^2 + B(r)\,dr^2 + r^2 d\Omega_2^2$$

$$\begin{cases} A(r) \xrightarrow[r\to\infty]{} 1 - \frac{2GM}{r} \\ B(r) \xrightarrow[r\to\infty]{} 1 \end{cases}$$

Vácuo $\to \boxed{R_{\mu\nu}=0}$

Não Vácuo $\to R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \neq 0$

$$R_{\nu\rho} = \left(\partial_\sigma \Gamma^\sigma_{\nu\rho} + \Gamma^\sigma_{\sigma\kappa}\Gamma^\kappa_{\nu\rho}\right) - \left(\partial_\nu \Gamma^\sigma_{\sigma\rho} + \Gamma^\sigma_{\nu\kappa}\Gamma^\kappa_{\sigma\rho}\right)$$

HW

$$\Gamma^t_{tr} = \frac{A'}{2A} \quad \Gamma^r_{tt} = \frac{A'}{2B} \quad \Gamma^r_{rr} = \frac{-B'}{2B} \quad \Gamma^r_{\theta\theta} = -\frac{r}{B} \quad \Gamma^r_{\varphi\varphi} = \frac{-r\sin^2\theta}{2B}$$

$$\Gamma^\theta_{r\theta} = \frac{1}{r} \quad \Gamma^\varphi_{r\varphi} = \frac{1}{r} \quad \Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta \quad \Gamma^\varphi_{\theta\varphi} = \cot\theta$$

os outros = 0

$$\boxed{R_{\mu\nu}=0}\quad \begin{cases} R_{tt} = \cdots = \dfrac{A''}{2B} + \dfrac{A'}{rB} - \dfrac{A'}{4B}\left(\dfrac{A'}{A}+\dfrac{B'}{B}\right) = 0 \\ \qquad (HW) \\ R_{rr} = \cdots = -\dfrac{A''}{2A} + \dfrac{B'}{4B}(\cdots) = 0 \\ R_{\theta\theta} = 1 - \dfrac{1}{B} - \dfrac{r}{2B}\left(\dfrac{A'}{A}-\dfrac{B'}{B}\right) \\ R_{\varphi\varphi} = r^2 \cancel{R_{\theta\theta}} = \cdots = 0 \end{cases}$$

3 equações de 2ª ordem para 2 funções $A(r)$ $B(r)$

Será que o sistema é sobredeterminado?

Não → identidade de Bianchi (ver depois)

$$\frac{R_{tt}}{A} + \frac{R_{rr}}{B} = \frac{1}{rB}\underbrace{\left(\frac{A'}{A}+\frac{B'}{B}\right)}_{=0} = 0$$

$$\frac{d}{dr}\ln A + \frac{d}{dr}\ln B = 0 \quad \rightarrow \quad \frac{d}{dr}\ln(AB) = 0$$

$$\longrightarrow A.B = \text{const.} = 1$$

cond. a ∞

$$\boxed{B = \frac{1}{A}}$$

de $R_{\theta\theta}$:

$$r\left(\frac{1}{B}\right)' + \frac{1}{B} = 1 \longrightarrow \frac{1}{B} = 1 + \frac{\text{const}}{r} = A$$

$A = \dfrac{1}{B}$

$$\boxed{\text{const} = -2GM}$$

$$\boxed{R_{rr} = 0}$$

$$\Rightarrow \boxed{ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2GM}{r}} + r^2 d\Omega_2^2}$$

Válida para $\boxed{r > R}$ ↳ raio da estrela.

Métrica de Schwarzchild

$$\boxed{r_s = 2GM} \begin{cases} g_{tt} \to 0 \\ g_{rr} \to \infty \end{cases}$$ Singularidade na métrica!

Singularidade de curvatura vs. Sing. de coordenadas

Escalar de Curvatura : $\underbrace{R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}}_{\text{Escalar de Kretschmann}}$

→ para a métrica de Schwarzchild

$$R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = \left.\frac{12(2GM)^2}{r^6}\right|_{r=r_s} \text{ finito!}$$

$(c \neq 1)$

$\Rightarrow r_s = \frac{2GM}{c^2}$ é uma singularidade de coordenadas!

$M_{Sol} = 2 \times 10^{30}\, Kg$

$M_{Terra} = 6 \times 10^{24}\, Kg$

$G_N = 6,7 \times 10^{-11} \frac{m^3}{Kg\, s^2}$

$c = 3 \times 10^8\, m/s$

$$\Rightarrow \left.\begin{aligned} r_s|_{Terra} &= \dots = 1\, cm \\ r_s|_{Sol} &= \dots = 3\, Km \end{aligned}\right]$$ não precisamos nos preocupar com a singularidade em r_s pois $r_s < R$

Constante Cosmológica

$$\int d^4x \sqrt{-g} \times \text{escalar}$$

$\downarrow$

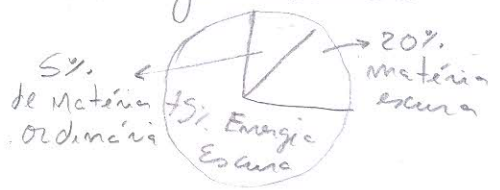
Pode ser um número puro

$$S_{EH} + S_{cosmo} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \, (R - 2\Lambda)$$

↳ constante cosmológica

$\downarrow$

Energia Escura



$$\delta(S_{EH} + S_{cosmo}) = 0$$

$$0 = \delta(\sqrt{-g})(R - 2\Lambda) + \sqrt{-g}\,\delta R$$

$\vdots$ (HW)

$$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0}$$

Uma constante muda as equações do movimento devido à $\sqrt{-g}$!