Relatividade: 11/05

lista 5

5) $T_{\mu\nu} = F_{\mu\sigma}F_{\nu}{}^{\sigma} - \frac{1}{4} g_{\mu\nu} F_{\sigma\tau} F^{\sigma\tau}$

mostrar $D_\mu T^{\mu\nu} = 0$

$A_{\mu\nu} = B_\mu C_\nu$

$A^{\mu\nu} = B^\mu C^\nu$

$A^{\mu\nu} = g^{\mu\rho} g^{\nu\sigma} A_{\rho\sigma}$

$= g^{\mu\rho} B_\rho g^{\nu\sigma} C_\sigma$

a)

$\Rightarrow T^{\mu\nu} = F^{\mu}{}_{\sigma} F^{\nu\sigma} - \frac{1}{4} g^{\mu\nu} F_{\sigma\tau} F^{\sigma\tau}$

$$\begin{cases} D_\mu F^{\mu\nu} = 0 \\ D_\mu F_{\rho\sigma} + D_\rho F_{\sigma\mu} + D_\sigma F_{\mu\rho} = 0 \end{cases}$$

$D_\mu T^{\mu\nu} = D_\mu \left(\underbrace{F^{\mu}{}_{\sigma} F^{\nu\sigma}}_{①} - \frac{1}{4} g^{\mu\nu} \underbrace{F_{\sigma\tau} F^{\sigma\tau}}_{②} \right)$

① $- (D_\mu F^{\mu}{}_{\sigma}) F^{\nu\sigma} + F^{\mu\sigma} (D_\mu F^{\nu}{}_{\sigma})$

$D_\mu F^{\mu}{}_{\sigma} = 0$

$\rightarrow F^{\mu\sigma} D_\mu F^{\nu}{}_{\sigma}$

$= F^{\mu\sigma} D_\mu g^{\nu\lambda} F_{\lambda\sigma}$

$= F^{\mu\sigma} g^{\nu\lambda} D_\mu F_{\lambda\sigma}$

$= \frac{1}{2} F^{\mu\sigma} g^{\nu\lambda} D_\mu F_{\lambda\sigma} + \frac{1}{2} F^{\mu\sigma} g^{\nu\lambda} D_\mu F_{\lambda\sigma}$ ($F^{\mu\sigma} = -F^{\sigma\mu}$, $\mu \leftrightarrow \sigma$)

$= \frac{1}{2} F^{\mu\sigma} g^{\nu\lambda} D_\mu F_{\lambda\sigma} - \frac{1}{2} F^{\mu\sigma} g^{\nu\lambda} D_\sigma F_{\lambda\mu}$

$= \frac{1}{2} F^{\mu\sigma} g^{\nu\sigma} \left[D_\mu F_{\lambda\sigma} + D_\sigma F_{\mu\lambda} \right]$

Id. Bian. $= - D_\lambda F_{\sigma\mu}$

① $= -\frac{1}{2} F^{\mu\sigma} g^{\nu\lambda} D_\lambda F_{\sigma\mu}$

$F^{\mu\sigma} D_\mu F^{\nu}{}_{\sigma}$

$D_\mu g_{\sigma\lambda} F^{\nu\lambda}$

$= F^{\mu\sigma} g_{\sigma\lambda} D_\mu F^{\nu\lambda}$

$= F^{\mu}{}_{\sigma} D_\mu F^{\nu\sigma}$

Fazer análogo para ②

① + ② = $\frac{1}{2} F^{\mu\beta} g^{\alpha\nu}$ (Bianchi) $= 0$

$$T_\mu{}^\nu = F_{\mu\sigma} F^{\nu\sigma} - \frac{1}{4} \delta_\mu{}^\nu F_{\sigma\tau} F^{\sigma\tau}$$

$$T_\mu{}^\mu = F_{\mu\sigma} F^{\mu\sigma} - \frac{1}{4} \delta^\mu{}_\mu \delta^\nu{}_\mu F_{\sigma\tau} F^{\sigma\tau}$$

b)

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0 \qquad S = S_{EH} + S_{maxwell}$$

↳ grav. pura

$$\frac{\delta S}{\delta g^{\mu\nu}} = \frac{\delta S_{EH}}{\delta g^{\mu\nu}} + \frac{\delta S_{maxwell}}{\delta g^{\mu\nu}} = 0$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \text{\#} \frac{1}{\sqrt{-g}} \frac{\delta \mathcal{L}_{maxwell}}{\delta g^{\mu\nu}} \propto T_{\mu\nu}$$

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_{maxwell}}{\delta g^{\mu\nu}}$$

$$S_{maxwell} = -\frac{1}{4} \int d^4x \sqrt{-g}\, F_{\mu\nu} F^{\mu\nu}$$

$F_{\mu\nu} g^{\mu\mu'} g^{\nu\nu'} F_{\mu'\nu'}$ → métricas escondidas

$$\mathcal{L}_{maxwell} = -\frac{1}{4} \sqrt{-g}\, F_{\mu\nu} F^{\mu\nu}$$

$$\delta\left(-\frac{1}{4} \underset{①}{\sqrt{-g}}\, \underset{②}{g^{\mu\alpha}}\, \underset{③}{g^{\nu\beta}} F_{\mu\nu} F_{\alpha\beta}\right) \qquad F^2$$

$$\delta\sqrt{-g} = -\frac{\sqrt{-g}}{2} g_{\mu\nu} \delta g^{\mu\nu} = \frac{\sqrt{-g}}{2} g^{\mu\nu} \delta g_{\mu\nu}$$

$$① = -\frac{1}{4}\left(-\frac{\sqrt{-g}}{2}\right) g_{\lambda\tau} \delta g^{\lambda\tau} F^2$$

$$② = -\frac{1}{4} \sqrt{-g}\, \delta g^{\mu\alpha} g^{\nu\beta} F_{\mu\nu} F_{\alpha\beta}$$

$$⑤ = -\frac{1}{4}\sqrt{-g}\, g^{\mu\alpha}_{\beta}\left(\delta g^{\nu\beta}\right) F_{\mu\nu} F_{\alpha\beta}$$

$$① + ② + ③ = \frac{1}{8}\sqrt{-g}\, g_{\mu\nu}\,\delta g^{\mu\nu} F^2 - \frac{1}{4}\sqrt{g}\Big(g^{\alpha\beta} \overset{(-1)\times(-1)}{f_{\mu\alpha}\, F_{\nu\beta}} + g^{\alpha\beta} F_{\beta\nu} F_{\alpha\mu}\Big)\delta g^{\mu\nu}$$

$$= \frac{1}{8}\sqrt{g}\, g_{\mu\nu}\delta g^{\mu\nu} F^2 - \frac{1}{2}\sqrt{-g}\, g^{\alpha\beta} F_{\alpha\mu} F_{\beta\nu}\delta g^{\mu\nu}$$

$$= \frac{\sqrt{-g}}{2}\delta g^{\mu\nu}\left(\frac{1}{4} g_{\mu\nu} F^2 - F^{\alpha}{}_{\mu} F_{\alpha\nu}\right)$$

$$\frac{\delta \mathcal{L}_{EM}}{\delta g^{\mu\nu}} = -\frac{\sqrt{-g}}{2}\underbrace{\left(F_{\alpha\mu} F^{\alpha}{}_{\nu} - \frac{1}{4} g_{\mu\nu} F^2\right)}_{= T_{\mu\nu}}$$

$$\hookrightarrow = -\frac{\sqrt{-g}}{2} T_{\mu\nu}$$

↳ Reissner-Nordström

1. $D_\mu T^{\mu\nu} \qquad T^{\mu\nu} = A^\mu B^\nu$

$$= \underbrace{(D_\mu A^\mu)}_{\text{"}\partial+\Gamma\text{"}} B^\nu + A^\mu \underbrace{D_\mu B^\nu}_{\text{"}\partial+\Gamma\text{"}}$$

$$= (\partial_\mu A^\mu) B^\nu + A^\mu \partial_\mu B^\nu + \Gamma\ldots + \Gamma\ldots$$

$$\partial_\mu (A^\mu \cdot B^\nu) + \Gamma\ldots + \Gamma\ldots$$