Relatividade - Lista 7 de exercícios

1) $ds^2 = -A(r)\,dt^2 + B(r)\,(dr^2 + r^2 d\Omega_2^2)$

$\qquad\qquad\qquad\qquad = dx^2 + dy^2 + dz^2$

fator conforme

$\lambda(\rho) \qquad \rho = \rho(r)$

$$\begin{cases} ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + [\lambda(\rho)]^2 (d\rho^2 + \rho^2 d\Omega_2^2) \\ \quad = -\left(1 - \frac{2GM}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2GM}{r}} + r^2 d\Omega_2^2 \end{cases}$$

$$\begin{cases} [\lambda(\rho)]^2 \rho^2 = r^2 \\ \frac{dr^2}{1 - \frac{2GM}{r}} = [\lambda(\rho)]^2 d\rho^2 \end{cases}$$

$\pm \frac{d\rho}{\rho} = \frac{dr}{\sqrt{r^2 - 2GM}}$

$\Rightarrow r \to \infty \Longrightarrow \rho \to \infty$

$\to +$

então $\boxed{r = \rho\left(1 + \frac{GM}{2\rho}\right)^2}$

$$\Rightarrow [\lambda(\rho)]^2 = \left(1 + \frac{GM}{2\rho}\right)^4 \Rightarrow ds^2 = -\frac{\left(1 - \frac{GM}{2\rho}\right)^2}{\left(1 + \frac{GM}{2\rho}\right)^2} dt^2 + \left(1 + \frac{GM}{2\rho}\right)^4 (d\rho^2 + \rho^2 d\Omega_2^2)$$

2) Cap VII . 1 Zee

a) $ds^2 = -A(r)\,dt^2 + B(r)\,dr^2 + r^2 d\Omega_2^2$

Massa (m)

$$L = \left(A(r)\left(\frac{dt}{d\tau}\right)^2 - B(r)\left(\frac{dr}{d\tau}\right)^2 - r^2\left(\frac{d\theta}{d\tau}\right)^2 - r^2\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2\right)^{1/2}$$

$$= \sqrt{-\frac{ds^2}{d\tau^2}}$$

para $ds^2 > 0$

$*$, $\cancel{*}$ $\rightarrow$ 2 constante do movimento

$(\tau) \rightarrow \sqrt{} = 1$

$$\begin{cases} \dfrac{d}{d\tau}\left(A(r)\dfrac{dt}{d\tau}\right) = \dfrac{dL}{dt} = 0 & (*) \\ \dfrac{d}{d\tau}\left(B(r)\dfrac{dr}{d\tau}\right) + \dfrac{1}{2}A'\left(\dfrac{dt}{d\tau}\right)^2 - \dfrac{1}{2}B'\left(\dfrac{dr}{d\tau}\right)^2 - r\left(\dfrac{d\theta}{d\tau}\right)^2 + \\ \qquad - r\sin^2\theta\left(\dfrac{d\phi}{d\tau}\right)^2 = 0 \\ \dfrac{d}{d\tau}\left(r^2\dfrac{d\theta}{d\tau}\right) - r^2\sin\theta\cos\theta\left(\dfrac{d\phi}{d\tau}\right)^2 = 0 \\ \dfrac{d}{d\tau}\left(r^2\sin^2\theta\dfrac{d\phi}{d\tau}\right) = \dfrac{dL}{d\phi} = 0 & (**) \end{cases}$$

$$\begin{cases} A(r)\dfrac{dt}{d\tau} = \text{const} = \epsilon & (*) \\ r^2\sin^2\theta\dfrac{d\phi}{d\tau} = \text{const} = \ell & (**) \end{cases}$$

Sem massa

$$S = -m\int ds \longrightarrow \tilde{S} = \frac{1}{2}\int d\tau\left(e^{-1}\dot{x}^2 - m^2 e\right)$$

$e \to$ campo auxiliar.

3. $(G=1)$

$R_S = 2M$ — $r = 10M$

radial $\to$ $\underline{\ell = 0}$

$$\epsilon = \left(1 - \frac{2M}{r}\right)u^t\Big|_{r=10M}$$

$$u^t = \left(1 - \frac{2M}{r}\right)^{-1/2}\Big|_{r=10M} = \frac{\sqrt{5}}{2}$$

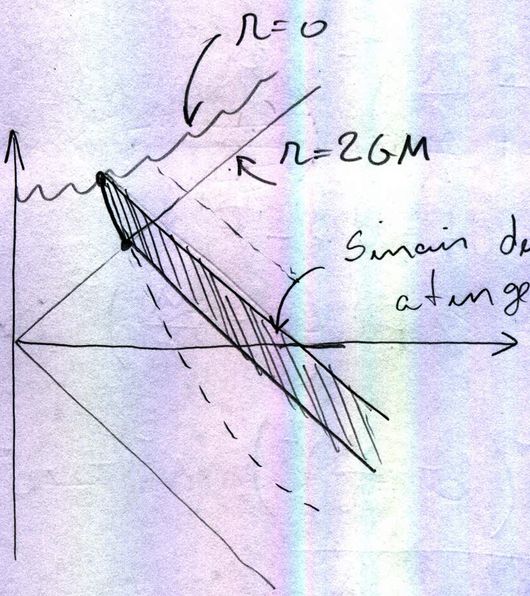
Usando $u \cdot u = -1$ (4-velocidade) $\longrightarrow$ $\left(\frac{dr}{d\tau}\right)^2 = \epsilon^2 - \left(1 - \frac{2M}{r}\right)$

$$= \frac{2M}{r} - \frac{1}{5}$$

$$\tau = \int_0^{10M} dr\left(\frac{2M}{r} - \frac{1}{5}\right) \xrightarrow{r = 10M\xi} \tau = 10\sqrt{5}M\int_0^1 d\xi\left(\frac{1}{\xi} - 1\right)^{-1/2}$$

$$= 5\sqrt{5}\,\pi M$$

4) KS

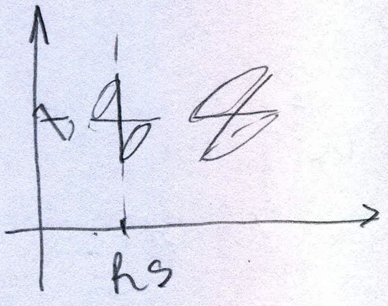


5) $\frac{dr}{dv} = \frac{1}{2}\left(1 + \frac{2|M|}{r}\right) > 0 \Rightarrow$ raios de luz são sempre outgoing

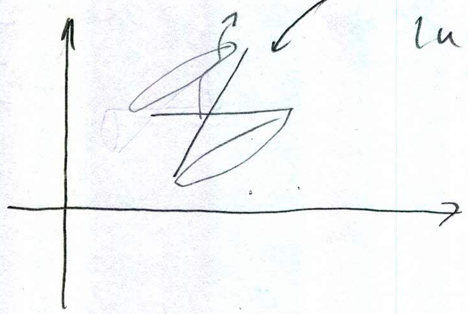
$|M| > 0$

A cada ponto tem um raio de luz <u>outgoing</u>

$v = \text{const} \rightarrow$ ingoing



Buraco Negro



o cone de luz é sempre assim

6) Feito na aula

7) Mathematica

8) Mathematica $\longrightarrow$ Introduzir const-cosmológica

$\Lambda = \frac{\#}{L^2}$ $\quad \# = -6$

9) (Schw.) $\rightarrow$ (tartaruga) $\rightarrow$ (E-F) $\rightarrow$ (K.-S)

10) a) $ds^2 = -\left(1 - \frac{M}{r}\right)^2 dt^2 + \left(1 - \frac{M}{r}\right)^{-2} dr^2 + r^2 d\Omega_2^2$

G=1

$t = v - F(r) \longrightarrow dt = dv - F' dr$, $\quad F' = \frac{dF}{dr}$

$$ds^2 = -\left(1 - \frac{M}{r}\right)^2 dv^2 + 2F'\left(1 - \frac{M}{r}\right)^2 dv\,dr + \underbrace{\left(\left(1 - \frac{M}{r}\right)^{-2} - (F')^2\left(1 - \frac{M}{r}\right)^2\right)}_{=0} dr^2 + r^2 d\Omega_2^2$$

$$\boxed{F' = \left(1 - \frac{M}{r}\right)^{-2}}$$

$$= -\left(1 - \frac{M}{r}\right)^2 dv^2 + 2\,dv\,dr + r^2 d\Omega_2^2$$

não singular em $r = M$ (métrica fica não degenerada $\det \neq 0$)

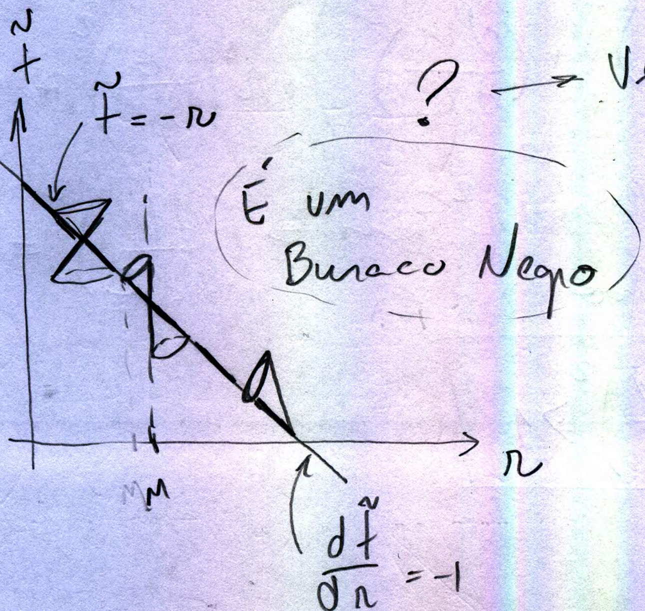
b) Integrar:

$$F' = \left(1 - \frac{M}{r}\right)^{-2} \longrightarrow t = v - (r - M) - 2M \ln\left|\frac{r}{M} - 1\right| + \frac{M^2}{r - M}$$

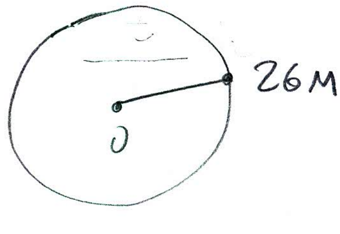
$$t = r = 0 : v = 0$$

Cone de luz $\begin{cases} v = \text{const.} \\ \dfrac{dv}{dr} = \dfrac{2}{\left(1 - \frac{M}{r}\right)^2} \end{cases}$ ou para $\tilde{t} = v - r$ $\begin{cases} \dfrac{d\tilde{t}}{dr} = -1 \\ \dfrac{d\tilde{t}}{dr} = -1 + \dfrac{2}{\left(1 - \frac{M}{r}\right)^2} \end{cases}$

? $\longrightarrow$ Verificar



11) $\tau = -\displaystyle\int_{26M}^{0} \frac{dr}{\left(\frac{dr}{d\tau}\right)}$



$$\overset{G=1}{=} -\int_{2M}^{0} dr \left(\epsilon^2 - \left(1+\frac{\ell^2}{r^2}\right)\left(1-\frac{2M}{r}\right)\right)^{-\frac{1}{2}}$$

Verificar

$$= \int_{0}^{2M} dr \left(\epsilon^2 + \left(1+\frac{\ell^2}{r^2}\right)\left(\frac{2M}{r}-1\right)\right)^{-1/2}$$

Numerador menor possível $\epsilon = 0$, $\ell = 0$

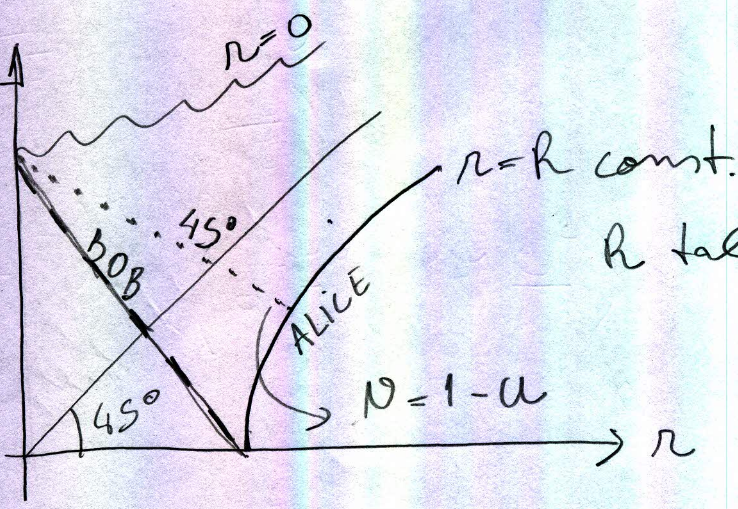
$$= \int_{0}^{2M} dr \left(\frac{2M}{r}-1\right)^{-1/2} = 2M \int_{0}^{1} \frac{\sqrt{\xi}}{\sqrt{1-\xi}} = \pi M$$

12) trajetória de tipo-tempo $u \cdot u = 1$

(G=1) $$\frac{32M^3 e^{-\frac{r}{2M}}}{r}\underbrace{\left(-\left(\frac{dv}{d\tau}\right)^2+\left(\frac{du}{d\tau}\right)^2\right)}_{\text{tem que ser negativo}} + r^2\left(\frac{d\theta}{d\tau}\right)^2 + r^2\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2 = -1$$

$$\left(\frac{dv}{d\tau}\right)^2 > \left(\frac{du}{d\tau}\right)^2 \Longrightarrow \boxed{\left(\frac{dv}{d\tau}\right)^2 > 1}$$

13) a)



R tal que $\sqrt{\frac{R}{2M}-1}\; e^{\frac{R}{4M}} = \frac{1}{2}$

b) BOB → linha de mundo tem pendência = +2

⇒ dentro do cone de luz

⇒ trajetória de tipo tempo

c) t máximo = t no qual a linha de 45° desde $u=0$, $v=1$ intersecta $r=R$

$(t,r)\to(u,v)$

$$\hookrightarrow \left(\frac{R}{2M}-1\right)^{1/2} e^{\frac{R}{4M}}\, \mathrm{sh}\left(\frac{t}{4M}\right) = 1 + \left(\frac{R}{2M}-1\right)^{1/2} e^{\frac{R}{4M}}\, \mathrm{ch}\left(\frac{t}{4M}\right)$$

$\Rightarrow t = 4M \ln 2$

Livro Sean Carroll → TOV

↳ arxiv.org