Resolução da P3 20/6/2016

1) $U \cdot U = -1$
invariante

repouso $\Rightarrow u^i = 0$

$-1 = g_{\mu\nu} u^\mu u^\nu = g_{tt} u^t u^t$

$\Rightarrow u^t = \sqrt{-\frac{1}{g_{tt}}} = \sqrt{\frac{1}{1-\frac{2GM}{r}}}$

2) $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$

$\hookrightarrow T_\mu{}^\mu = 0$

$g^{\mu\nu}\left(R_{\mu\nu} - g_{\mu\nu} R = 8\pi G T_{\mu\nu}\right)$

$R - \frac{4}{2} R = 8\pi G T = 0 \Rightarrow R = 0$

$T_{\mu\nu} = \underbrace{F_{\mu\sigma} F_\nu{}^\sigma} - \frac{1}{4} g_{\mu\nu} F_{\sigma\tau} F^{\sigma\tau}$

$T = F^2 - \frac{1}{4} 4 F^2 = 0$

3) a) $f(r_H) = 0$

$1 + r_H^2 - \frac{\mu}{r_H^2} = 0$

raiz positiva, real

$r_H = +\sqrt{\frac{\sqrt{4\mu+1} - 1}{2}}$

b) $L = \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}}$

$= \sqrt{f(r)\dot{t}^2 - \frac{1}{f(r)}\dot{r}^2 - r^2\dot{\phi}^2}$

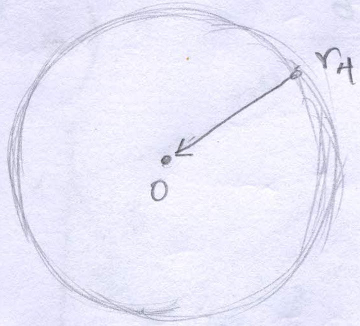
$-g_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -g_{\mu\nu} u^\mu u^\nu = \begin{cases} 1 & \text{tipo-tempo} \\ 0 & \text{nula, tipo-luz} \end{cases}$

$\dot{} = \frac{d}{dt}$

$\begin{cases} \mathcal{E} = f(r)\dot{t} = cte \\ J = r^2\dot{\phi} = cte \end{cases}$

$\begin{cases} \text{mov. radial} \rightarrow J = 0 \\ -g_{\mu\nu} u^\mu u^\nu = 1 \end{cases}$

$1 = \frac{\mathcal{E}^2}{f(r)} - \frac{\dot{r}^2}{f(r)} \Rightarrow \dot{r}^2 = \mathcal{E}^2 - f(r)$



$\dot{r} = 0$

$f(r_H) = 0 \longrightarrow \boxed{\mathcal{E} = 0}$

$$\frac{dr}{d\tau} = \sqrt{-f(r)} \longrightarrow \tau = \int_0^{r_H} \frac{dr}{\sqrt{f(r)}}$$

$$= \frac{\pi}{4} - \frac{1}{2}\cot^{-1}(2\sqrt{\mu})$$

c) $\begin{cases} J \neq 0 \\ -g_{\mu\nu} u^\mu u^\nu = 0 \end{cases}$

$$\frac{1}{2}\dot{r}^2 + \underbrace{\frac{1}{2} f(r) \frac{J^2}{r^2}}_{} = \frac{1}{2}\mathcal{E}^2$$

$$\rightarrow V_{eff} = \frac{J^2}{2r}\left(1 + r^2 - \frac{\mu}{r^2}\right)$$

$\rightarrow$ @ $r = \sqrt{2\mu}$:

$$\mathcal{E}^2 = f(\sqrt{2\mu})\frac{J^2}{2\mu} \Rightarrow \frac{J^2}{\mathcal{E}^2} = \frac{4\mu}{4\mu + 1}$$

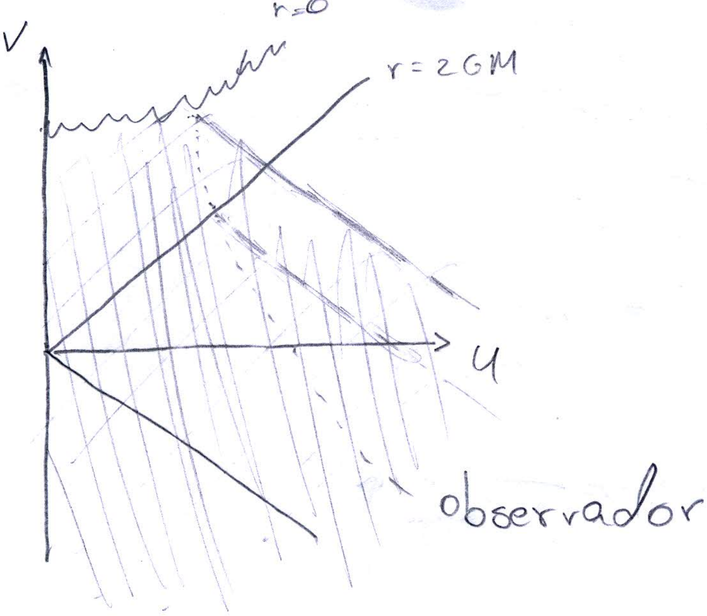
"turning point": $-\frac{J^2}{r^3} + 2\mu \frac{J^2}{r^5} = 0$

$r = \sqrt{2\mu} \longrightarrow$ geodésica nula com $r = cte$

$$\frac{d\varphi}{dt} = \frac{d\varphi}{d\xi}\frac{d\xi}{dt} = \frac{J}{r^2}\frac{f(r)}{\mathcal{E}} = \frac{1}{2}\sqrt{\frac{4\mu + 1}{\mu}} \Rightarrow t_p = 4\pi\sqrt{\frac{\mu}{4\mu+1}}$$

período

5)



6) $\rho = cte$

TOV1: $M(r) = \frac{4}{3}\pi r^3 \rho$

$M = 0$ @ $r = 0$

TOV2: $\begin{cases} \dfrac{-dP}{(\rho + P)(\frac{\rho}{3} + P)} = \dfrac{4\pi r^3 dr}{r^2\left(1 - \frac{8\pi\rho r^2}{3}\right)} \\ P(R) = 0 \end{cases}$

$$\Rightarrow P(r) = \rho \left(\frac{(1 - 8\pi\rho r^2/3)^{1/2} - (1 - 8\pi\rho R^2/3)^{1/2}}{3(1 - 8\pi\rho R^2/3)^{1/2} - (1 - 8\pi\rho r^2/3)^{1/2}} \right)$$

$$P(r=0) \sim \frac{\#}{3(1 - 8\pi\rho R^2/3)^{1/2}}$$

Massa total: $M = \frac{4}{3}\pi\rho R^3$

$R = R_{max}$ a pressão no centro $\longrightarrow \infty$

R_{max} : $1 - \frac{8}{3}\pi \rho R_{max}^2 = 0$

$$R_{max} = \sqrt{\frac{3}{8}\frac{1}{\pi\rho}}$$

$$M_{max} = \frac{4}{3}\pi\rho R_{max}^3$$

b) $P = \rho = \frac{\alpha}{r^n}$

TOV1) $\frac{dm}{dr} = 4\pi\alpha r^{2-n} \longrightarrow m = \frac{4\pi\alpha}{3-n} r^{3-n}$

TOV2) $\frac{dP}{dr} = -\frac{n\alpha}{r^{n+1}} = -\frac{2\alpha}{r^n} \frac{\frac{4\pi\alpha}{3-n} r^{3-n} + 4\pi\alpha r^{3-n}}{r\left(r - \frac{8\pi\alpha}{3-n} r^{3-n}\right)}$

$$n = 2\frac{\left(\frac{4\pi\alpha}{3-n} + 4\pi\alpha\right) r^{3-n}}{\left(r - \frac{8\pi\alpha}{3-n} r^{3-n}\right)} \longrightarrow \boxed{n = 2}$$

$n = 2$

$\alpha = \frac{1}{16\pi}$ $\quad P = \rho = \frac{1}{16\pi r^2}$ $\quad m = \frac{r}{4}$

$$\begin{cases} M = \int_0^R 4\pi r^2 \rho \, dr = \frac{R}{4} \\ P = \frac{1}{16\pi R^2} \end{cases}$$

Continuação da matéria

$$\begin{cases} \text{matéria: } \rho \propto a^{-3} \\ \text{Radiação: } \rho \propto a^{-4} \\ \text{EE: } \rho \propto a^{0} \end{cases}$$

$$\begin{cases} w = \dfrac{P}{\rho} \begin{cases} w = 0 & \text{matéria} \\ w = \frac{1}{3} & \text{radiação} \\ w = -1 & \text{EE} \end{cases} \\ \rho \propto a^{-3(1+w)} \end{cases}$$

Eq. de Einstein

$$\begin{cases} R_H = -3\dfrac{\ddot{a}}{a} \qquad R_{ij} = \left(\dfrac{\ddot{a}}{a} + 2\left(\dfrac{\dot{a}}{a}\right)^2 + \dfrac{2K}{a^2}\right)a^2 \tilde{g}_{ij} \\ R = -6\left(\dfrac{\ddot{a}}{a} + \left(\dfrac{\dot{a}}{a}\right)^2 + \dfrac{k}{a^2}\right) \end{cases}$$

$$\overset{HW}{\Longrightarrow} \begin{cases} \left(\dfrac{\dot{a}}{a}\right)^2 = \dfrac{8\pi G}{3}\rho - \dfrac{k}{a^2} = H^2 \\ \dfrac{\ddot{a}}{a} = -\dfrac{4\pi G}{3}(\rho + 3P) \end{cases}$$

Equações de Friedmann

$$\begin{cases} t_0 = \text{today} \\ k = 0 \text{ Univ. plano} \end{cases}$$

$$\rho_{crit,0} = \frac{3H_0^2}{8\pi G} = 1{,}9 \times 10^{-29}\, h^2 \cdot g/cm^3$$

$h = 0{,}71 \pm 0{,}03$

Parâmetros adimensionais

$$\Omega_{i,0} = \frac{\rho_{i,0}}{\rho_{crit,0}} \qquad i = \begin{cases} \text{matéria} \to m \\ \text{radiação} \to r \\ \text{EE} \to \Lambda \end{cases}$$

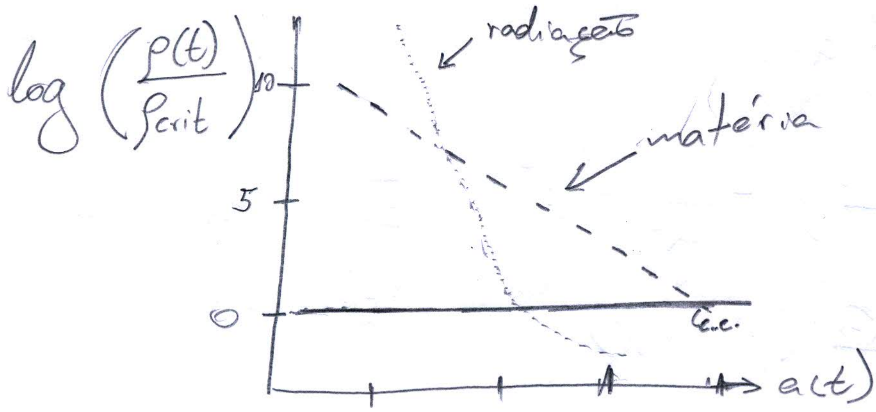
$$\Rightarrow H^2(a) = H_0^2 \left(\Omega_{r,0} \left(\frac{a_0}{a}\right)^4 + \Omega_{m,0} \left(\frac{a_0}{a}\right)^3 + \Omega_{k,0} \left(\frac{a_0}{a}\right)^2 + \Omega_{\Lambda,0} \left(\frac{a_0}{a}\right)^0 \right)$$

$$\Omega_{k,0} \equiv \frac{-k}{(a_0 H_0)^2}$$

$a_0 = 1$ + drop the 0's

$$\boxed{\frac{H^2}{H_0^2} = \Omega_r a^{-4} + \Omega_m a^{-3} + \cancel{\Omega_k} a^{-2} + \Omega_\Lambda}$$

obs: $|\Omega_k| \leq 0{,}01 \rightarrow \Omega_k = 0$



Universo com 1 síngula componente

$$\begin{cases} \rho \propto a^{-3(1+w)} \\ \frac{\dot{a}}{a} = \frac{8\pi G}{3}\rho - \cancel{\frac{k}{a^2}} \quad k=0 \end{cases}$$

$$\rightarrow \frac{\dot{a}}{a} \propto a^{-\frac{3}{2}(1+w)} \rightarrow a(t) \propto \begin{cases} t^{\frac{2}{3(1+w)}} & w \neq -1 \\ e^{Ht} & w = -1 \end{cases}$$

rad: $a \sim t^{1/2}$

mat: $a \sim t^{2/3}$

c.c.: $a \sim e^{Ht} \rightarrow$ exponencial

Universo com 2 componentes

matéria + radiação

$' : \frac{d}{d\tau} \leftarrow$ tempo conforme

$$\begin{cases} (a')^2 = \frac{8\pi G}{3} \rho a^4 \\ a'' = \frac{4\pi G}{3} (\rho - 3p) a^3 \end{cases}$$

Eq. de Friedmann em tempo conforme

$\hookrightarrow$ radiação contribue aqui pois $\rho_r - 3P_r = 0$

$$\rho = \rho_m + \rho_r = \frac{\rho_{eq}}{2}\left[\left(\frac{a_{eq}}{a}\right)^3 + \left(\frac{a_{eq}}{a}\right)^4\right]$$

$$a_{eq} = \frac{\Omega_r}{\Omega_m} \simeq 3\times 10^{-4}$$

$$\rho_m a^3 = cte = \frac{1}{2}\rho_{eq} a_{eq}^3$$

$$\Rightarrow a'' = \frac{2\pi G}{3}\rho_{eq} a_{eq}^3$$

$$\Rightarrow a(\tau) = \frac{\pi G}{3}\rho_{eq} a_{eq}^3 \tau^2 + c\tau + D$$

$$a(\tau=0) = 0 \rightarrow D = 0$$

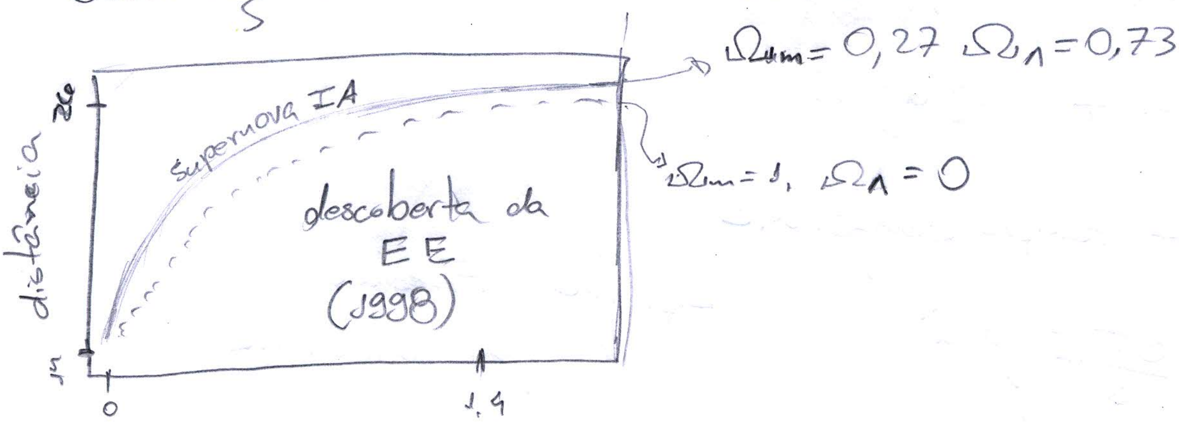
$$* \rightarrow c = \sqrt{\frac{4\pi G}{3}\rho_{eq} a_{eq}^4}$$

$$\Rightarrow a(\tau) = a_{eq}\left[\left(\frac{\tau}{\tau_*}\right)^2 + 2\left(\frac{\tau}{\tau_*}\right)\right]$$

$$\tau_* = \left(\frac{\pi G}{3}\rho_{eq} a_{eq}^2\right)^{-1/2} = \frac{\tau_{eq}}{\sqrt{2}-1}$$

$$\Rightarrow \begin{cases} \tau \ll \tau_{eq} : a \propto \tau \text{ (rad)} \\ \tau \gg \tau_{eq} : a \propto \tau^2 \text{ (matéria)} \end{cases}$$

Observações



ΛCDM

$$\begin{cases} |\Omega_k| \lesssim 0,01 \\ \Omega_r = 9,6 \times 10^{-5} \\ \Omega_{baryons} = 0,05 \\ \Omega_{Cold\ Dark\ Matter} = 0.27 \\ \Omega_\Lambda = 0,68 \end{cases}$$