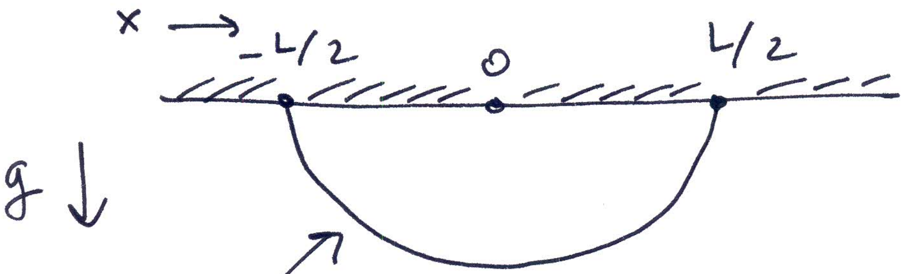
VARIATIONAL CALCULUS

= derivatives w.r.t. a function

Ex: elastic string hanging on a gravitational field



T: tension

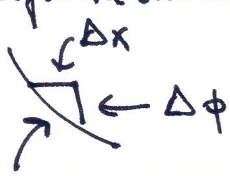
σ: linear mass density

$\phi(x)$

profile of the string as a function of x

w/ b.c. $\phi(\pm L/2) = 0$

infinitesimally



$$\sqrt{\Delta x^2 + \Delta \phi^2} = \Delta x \sqrt{1 + \left(\frac{\Delta \phi}{\Delta x}\right)^2}$$

→ length : $\displaystyle\int_{-L/2}^{L/2} dx \sqrt{1 + \left(\frac{d\phi}{dx}\right)^2}$

Two energies: elastic + gravitational

$$\text{elastic energy} = T \cdot \left(\text{length of extended string} - \text{intrinsic length } L \right)$$

$$= T \cdot \left(\int_{-L/2}^{L/2} dx \sqrt{1 + \left(\frac{d\phi}{dx}\right)^2} - L \right)$$

$$\simeq T \int_{-L/2}^{L/2} dx \, \frac{1}{2} \left(\frac{d\phi}{dx}\right)^2$$

$\frac{d\phi}{dx} \ll 1$

gravitational energy:

$$\int_{-L/2}^{L/2} dx \left(- \sigma \, g \, \phi(x) \right)$$

mass/unit length

− sign: $\phi(x)$ points downward

$\downarrow$

<u>Energy</u> <u>functional</u> = function of $\phi(x)$

$$E[\phi] = \int_{-L/2}^{L/2} dx \left(\frac{T}{2} \left(\frac{d\phi}{dx} \right)^2 - \sigma g \phi \right)$$

$E[\cdot]$ = function of a function

<u>Goal</u>: Find the function $\phi(x)$ which minimizes E $\rightarrow$ differentiate E w.r.t. ϕ

$$\phi(x) \longrightarrow \phi(x) + \eta(x)$$

small variation

$\boxed{\text{see later}}$ (in the end we'll call this $\delta\phi(x)$ but now I want to avoid possible confusions about "δ" commuting through derivatives, which it does!)

$$\Rightarrow \quad \delta E = E[\phi + \eta] - E[\phi]$$

It should: *) vanish at 1st order (= extremum)

*) > 0 at 2nd order (= minimum)

$$E[\phi+\eta] - E[\phi] = \int_{-L/2}^{L/2} dx \left(\frac{T}{2}\left(\frac{d\phi}{dx} + \frac{d\eta}{dx}\right)^2 - \sigma g(\phi+\eta) - \frac{T}{2}\left(\frac{d\phi}{dx}\right)^2 + \sigma g \phi \right)$$

$\boxed{\text{1st order in } \eta}$

$$\simeq \int_{-L/2}^{L/2} dx \left(T \frac{d\phi}{dx}\frac{d\eta}{dx} - \sigma g \eta \right)$$

$$= T \int_{-L/2}^{L/2} d\left(\eta \frac{d\phi}{dx}\right) - T \int_{-L/2}^{L/2} dx\, \eta \frac{d^2\phi}{dx^2} - \int_{-L/2}^{L/2} dx\, \sigma g \eta$$

by parts + $d\eta = \frac{d\eta}{dx} dx$

$$= \underbrace{T \frac{d\phi}{dx} \eta \Big|_{-L/2}^{L/2}}_{=0} - \underbrace{\int_{-L/2}^{L/2} dx \left(T \frac{d^2\phi}{dx^2} + \sigma g \right) \eta}_{= \delta E}$$

by (Dirichlet) boundary conditions $\eta(\pm L/2) = 0$

$\delta E = 0$ iff $T \frac{d^2\phi}{dx^2} = -\sigma g$ since η is generic

$$\Rightarrow \boxed{\phi(x) = \frac{\sigma g}{T}\left(\left(\frac{L}{2}\right)^2 - x^2\right)}$$ a parabola

such that $\phi(\pm L/2) = 0$

Now, shortcut without introducing η:

functional derivative $$\frac{\delta}{\delta\phi(y)}\int dx\, V[\phi(x)] = V'[\phi(y)]$$

$$E[\phi] = \int dx \underbrace{\left(F\left[\frac{d\phi}{dx}\right] + V[\phi]\right)}_{\equiv\ \mathcal{E}\left[\phi,\ \frac{d\phi}{dx}\right]}$$

$$\delta E[\phi] = \int dx \left(\frac{\delta F}{\delta \frac{d\phi}{dx}}\, \underbrace{\delta \frac{d\phi}{dx}}_{= \frac{d}{dx}\delta\phi} + \frac{\delta V}{\delta\phi}\,\delta\phi \right)$$

it is not an operation "δ" acting on ϕ

(remember $\delta\phi$ is just an η different from ϕ, so no issue in commuting δ & derivatives!)

$$= \int dx \left(\frac{d}{dx}\left(\frac{\delta F}{\delta\frac{d\phi}{dx}}\,\delta\phi\right) - \left(\frac{d}{dx}\frac{\delta F}{\delta\frac{d\phi}{dx}}\right)\delta\phi + \frac{\delta V}{\delta\phi}\,\delta\phi\right)$$

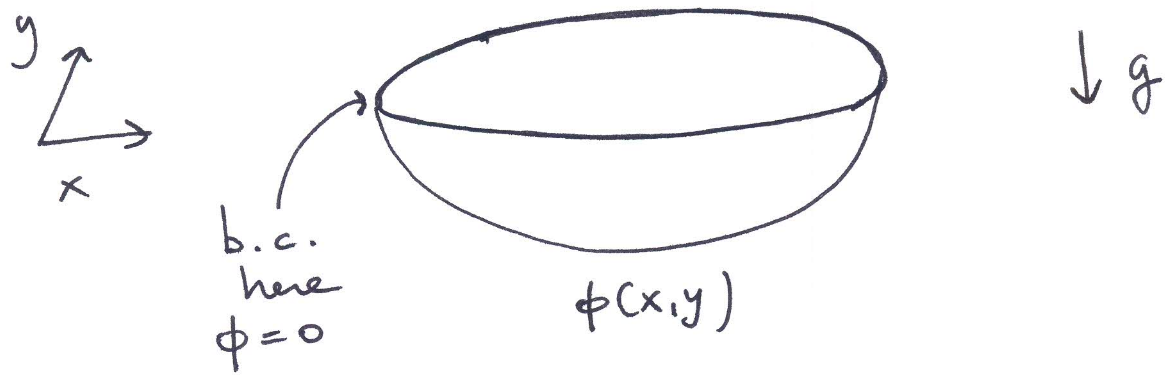
$$= \underbrace{\delta\phi\,\frac{\delta F}{\delta\frac{d\phi}{dx}}}_{=0}\Bigg|_{\text{bdry}} - \int dx\, \underset{\text{arbitrary}}{\delta\phi} \left(\frac{d}{dx}\frac{\delta F}{\delta\frac{d\phi}{dx}} - \frac{\delta V}{\delta\phi}\right)$$

$$\Rightarrow \boxed{\frac{d}{dx}\,\frac{\delta \mathcal{E}}{\delta \frac{d\phi}{dx}} - \frac{\delta \mathcal{E}}{\delta \phi} = 0}$$

Euler-Lagrange (E-L) equation

⋮

<u>Ex</u>: elastic <u>membrane</u> hanging on a gravit. field



$$E[\phi] = \int dx\, dy \left(\frac{\tilde{T}}{2}\left(\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 \right) - \rho(x,y)\,\phi(x,y) \right)$$

Apply same procedure as before: ($\tilde{T}$: membrane tension, ρ: load)

E-L equation = Poisson equation

$$\boxed{\nabla^2 \phi(x,y) = -\rho(x,y)}$$

$$\nabla^2 \phi(x,y) = (\partial_x^2 + \partial_y^2)\,\phi(x,y)$$

<u>Ex</u>: Newton's gravit. potential

$$\nabla^2 \phi(\underbrace{x,y,z}_{\equiv \vec{x}}) = 4\pi G\, \rho(x,y,z)$$

It emerges from the following energy functional

$$E[\phi] = \int d^3x \left(\frac{1}{8\pi G} (\vec{\nabla}\phi)^2 + \rho(\vec{x})\,\phi(\vec{x}) \right)$$

Check:

$$\delta E[\phi] = \frac{\delta E}{\delta \vec{\nabla}\phi}\,\delta\vec{\nabla}\phi + \frac{\delta E}{\delta\phi}\,\delta\phi =$$

$$= \int d^3\vec{x}\left(\frac{1}{4\pi G}\vec{\nabla}\phi\cdot\underbrace{\delta\vec{\nabla}\phi}_{=\vec{\nabla}\delta\phi} + \rho(\vec{x})\,\delta\phi\right)$$

$$= \underbrace{\int d^3\vec{x}\,\vec{\nabla}\cdot\left(\frac{1}{4\pi G}\vec{\nabla}\phi\,\delta\phi\right)}_{\longrightarrow 0 \text{ surface term}} - \int d^3\vec{x}\,\frac{1}{4\pi G}(\vec{\nabla}\cdot\vec{\nabla}\phi)\,\delta\phi + \int d^3x\,\rho(\vec{x})\,\delta\phi$$

$$\Rightarrow \boxed{\frac{1}{4\pi G}\vec{\nabla}^2\phi = \rho(\vec{x})} \quad \underline{\text{ok}}.$$

The same would be true for the electrostatic potential if ρ = charge density.

Bonus: use dimensional analysis to solve the equation above when $\rho(\vec{x}) = M\,\delta^3(\vec{x})$

$[\delta^3] = L^{-3}$ (as $\int d^3\vec{x}\,\rho(\vec{x}) = 1$)

$[\nabla^2] = L^{-2}$

$$\nabla^2\phi = 4\pi G M\,\delta^3(\vec{x}) \Rightarrow [\phi] = \frac{[GM]}{L}$$

By rotational invariance it must be

$$\phi = \frac{\alpha}{r} \quad \text{with } [\alpha] = [GM]$$

α: some constant

To determine the constant integrate both sides of the equation on a ball of radius R around the mass:

$$\int d^3\vec{x}\ \nabla^2\phi = \int d^3\vec{x}\ 4\pi G M\ \delta^3(\vec{x}) = 4\pi G M$$

$\parallel$ Gauss

$$\int d\vec{S}\cdot\vec{\nabla}\phi = \int_{r=R} r^2\, d\vartheta \sin\vartheta\, d\varphi\ \frac{\partial}{\partial r}\left(\frac{\alpha}{r}\right)$$

$$= R^2\, 4\pi\left(-\frac{\alpha}{R^2}\right)$$

$\Rightarrow \quad \alpha = -GM$

(Byproduct: $\nabla^2 \frac{1}{r} = -4\pi\,\delta^3(\vec{x})$.)

<u>Note</u>: One can consider more than one field:

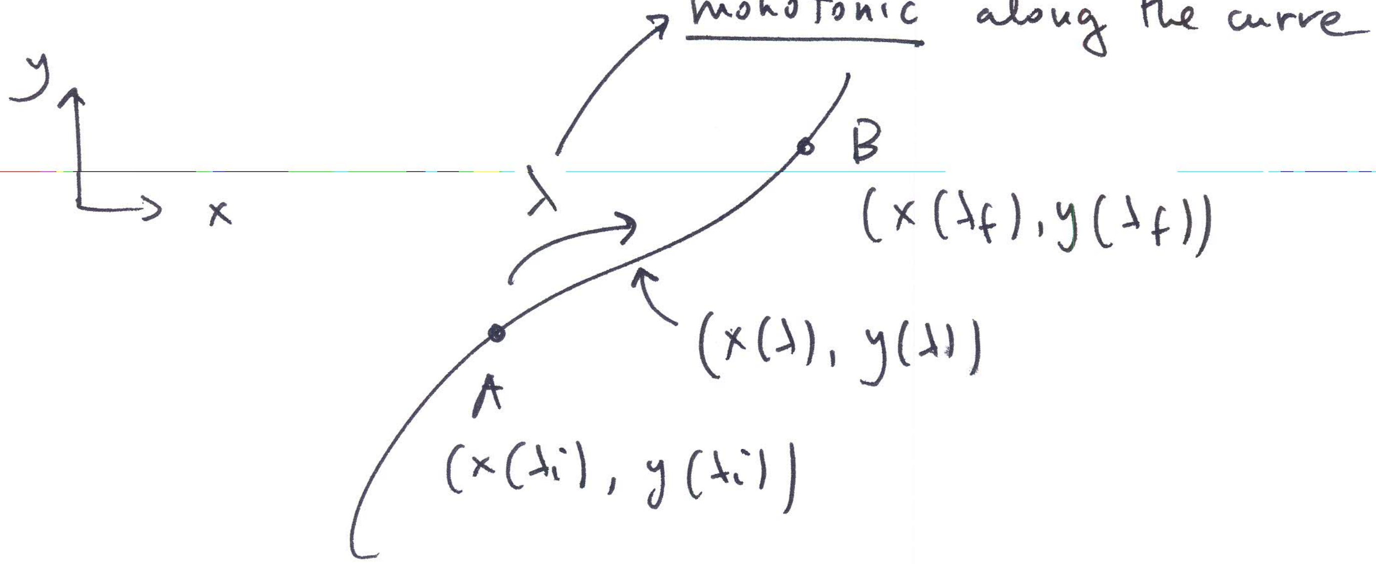
$$E[\phi_i] = \int dx\ \mathcal{E}\left[\phi_i, \frac{d\phi_i}{dx}\right]$$

$i = 1 \ldots \#$ of fields

$$\Rightarrow \quad \boxed{\frac{d}{dx}\left(\frac{\delta\mathcal{E}}{\delta\frac{d\phi_i}{dx}}\right) - \frac{\delta\mathcal{E}}{\delta\phi_i} = 0}$$

system of equations.

CURVES



length: $\displaystyle\int_A^B \sqrt{dx^2 + dy^2} = \int_{\lambda_i}^{\lambda_f} d\lambda \sqrt{\frac{dx^2}{d\lambda^2} + \frac{dy^2}{d\lambda^2}}$

obviously <u>reparametrization invariant</u>

also reparametriz. invariant

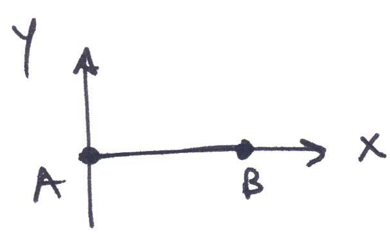
ex: $\lambda \to \alpha\lambda$ <u>OK</u>

$\lambda \to \eta(\lambda)$

$d\lambda = \dfrac{\partial\lambda}{\partial\eta}\, d\eta$ <u>OK</u>

<u>EX</u>: Straight line in $\mathbb{R}^2$

Imagine we don't know its equation & we want to find it via minimization



$\lambda = x$ $\qquad E = \displaystyle\int_0^x dx' \sqrt{1 + \left(\frac{dy}{dx'}\right)^2}$

length is minimized by

$$\frac{dy}{dx} = 0$$

This obvious result can be obtained in a more general way:

$$E(x,y) = \int d\lambda \; \mathcal{E}$$

$$\mathcal{E} = \sqrt{\left(\frac{dx}{d\lambda}\right)^2 + \left(\frac{dy}{d\lambda}\right)^2}$$

$\Rightarrow$ E-L equations:

$$\begin{cases} \dfrac{d}{d\lambda}\left(\dfrac{dx/d\lambda}{\sqrt{\left(\frac{dx}{d\lambda}\right)^2 + \left(\frac{dy}{d\lambda}\right)^2}} \right) = 0 \\[2ex] \dfrac{d}{d\lambda}\left(\dfrac{dy/d\lambda}{\sqrt{\left(\frac{dx}{d\lambda}\right)^2 + \left(\frac{dy}{d\lambda}\right)^2}} \right) = 0 \end{cases}$$

solution:

$$\begin{cases} \dfrac{dx}{d\lambda} = \text{const.} \\[2ex] \dfrac{dy}{d\lambda} = \text{const} \end{cases}$$

$\boxed{\text{Simplification:}}$ use $\lambda = l$ ← length of the curve

$$dl = \sqrt{dx^2 + dy^2}$$

$$\Rightarrow \sqrt{\left(\frac{dx}{d\lambda}\right)^2 + \left(\frac{dy}{d\lambda}\right)^2} = \frac{\sqrt{dx^2 + dy^2}}{dl} = 1$$

the square roots disappear!

$\Rightarrow$ the E-L equations become

$$\boxed{\frac{d^2x}{dl^2} = 0 = \frac{d^2y}{dl^2}}$$

In different coordinates systems this will look more complicated: let's look at polar coordinates

$$\int \sqrt{dr^2 + r^2 d\theta^2} = \int_{\lambda_i}^{\lambda_f} d\lambda \underbrace{\sqrt{\left(\frac{dr}{d\lambda}\right)^2 + r^2 \left(\frac{d\theta}{d\lambda}\right)^2}}_{\equiv L}$$

$$\left.\begin{aligned}\frac{d}{d\lambda}\left(\frac{\delta L}{\delta \frac{dr}{d\lambda}}\right) &= \frac{d}{d\lambda}\left(\frac{dr/d\lambda}{L}\right)\\ \frac{\delta L}{\delta r} &= \frac{r\,(d\theta/d\lambda)^2}{L}\end{aligned}\right\}\ \text{w.r.t. } r$$

$$\Rightarrow \boxed{\frac{d}{d\lambda}\left(\frac{dr/d\lambda}{L}\right) - \frac{r}{L}\left(\frac{d\theta}{d\lambda}\right)^2 = 0}$$

$$\left.\begin{aligned}\frac{d}{d\lambda}\left(\frac{\delta L}{\delta \frac{d\theta}{d\lambda}}\right) &= \frac{d}{d\lambda}\left(\frac{r^2\, d\theta/d\lambda}{L}\right)\\ \frac{\delta L}{\delta \theta} &= 0 \ \rightarrow \ \underline{\text{important!}}\end{aligned}\right\}\ \text{w.r.t. } \theta$$

$$\Rightarrow \boxed{\frac{d}{d\lambda}\left(\frac{r^2\, d\theta/d\lambda}{L}\right) = 0}$$

A convenient reparametrization is $\boxed{\lambda = \ell} \Rightarrow L = 1$

$$\boxed{\frac{d^2 r}{d\ell^2} - r\left(\frac{d\theta}{d\ell}\right)^2 = 0} \quad \& \quad \boxed{\frac{d}{d\ell}\left(r^2\frac{d\theta}{d\ell}\right) = 0}$$

Plus there is another (1st order) equation:

$$d\ell^2 = dr^2 + r^2\, d\theta^2 \Rightarrow \boxed{1 = \left(\frac{dr}{d\ell}\right)^2 + r^2\left(\frac{d\theta}{d\ell}\right)^2}$$

The two simplest equations to use are clearly this 1st order equation (= definition of l) & the equation for θ:

$$r^2 \frac{d\theta}{dl} = \text{const} \equiv a$$

$$\Longrightarrow \quad 1 = \left(\frac{dr}{dl}\right)^2 + r^2 \frac{a^2}{r^4} \quad \Longrightarrow \quad r^2 = l^2 + a^2$$

$$\Longrightarrow \quad \frac{d\theta}{dl} = \frac{a}{l^2 + a^2} \quad \Longrightarrow \quad \boxed{a \, \mathrm{tg}(\theta - \theta_0) = l}$$

= straight line in "funny" coordinates

<u>lesson</u>: a useful parametrization is the one that eliminates the square root

$$\underline{\underline{\lambda = l}} \qquad (\Longrightarrow L = 1)$$

<u>Ex</u>: "Geodesics" on a sphere

$$\int \sqrt{d\theta^2 + \sin^2\theta \, d\varphi^2} = \int_{\lambda_i}^{\lambda_f} d\lambda \underbrace{\sqrt{\left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2\theta \left(\frac{d\varphi}{d\lambda}\right)^2}}_{\equiv L}$$

$l = \lambda$:

$$\begin{cases} \dfrac{d^2\theta}{dl^2} = \sin\theta \cos\theta \left(\dfrac{d\varphi}{dl}\right)^2 \\[2ex] \dfrac{d}{dl}\left(\sin^2\theta \dfrac{d\varphi}{dl}\right) = 0 \quad \longrightarrow \quad \sin^2\theta \dfrac{d\varphi}{dl} = \text{const} \\[2ex] \left(\dfrac{d\theta}{dl}\right)^2 + \sin^2\theta \left(\dfrac{d\varphi}{dl}\right)^2 = 1 \end{cases}$$

(Killing vector)

Solution: $\varphi = \text{const}$ $\quad$ $(=$ great circles$)$

$\theta = \alpha l$

GEODESIC EQUATION

$X^\mu(\lambda)$ → parametrizes the curve

$g_{\mu\nu}(X)$ → metric of the space

$$\int dl = \int \sqrt{g_{\mu\nu}\, dX^\mu dX^\nu} = \int d\lambda \underbrace{\sqrt{g_{\mu\nu}(X(\lambda))\frac{dX^\mu}{d\lambda}\frac{dX^\nu}{d\lambda}}}_{\equiv L}$$

$$0 = \delta \int d\lambda\, L = \int d\lambda\, \delta L$$

$$= \int d\lambda \sqrt{g_{\mu\nu}(X+\delta X)\frac{d}{d\lambda}(X^\mu + \delta X^\mu)\frac{d}{d\lambda}(X^\nu + \delta X^\nu)}$$

$$- \int d\lambda \sqrt{g_{\mu\nu}(X)\frac{dX^\mu}{d\lambda}\frac{dX^\nu}{d\lambda}}$$

$$= \int d\lambda \left(\sqrt{\left(g_{\mu\nu}(X) + \partial_\sigma g_{\mu\nu}(X)\,\delta X^\sigma\right)\left(\frac{dX^\mu}{d\lambda} + \frac{d\delta X^\mu}{d\lambda}\right)\left(\frac{dX^\nu}{d\lambda} + \frac{d\delta X^\nu}{d\lambda}\right)}\right.$$

$$\left. - \sqrt{g_{\mu\nu}(X)\frac{dX^\mu}{d\lambda}\frac{dX^\nu}{d\lambda}}\right)$$

$$= \int d\lambda \left(\sqrt{g_{\mu\nu}(X)\frac{dX^\mu}{d\lambda}\frac{dX^\nu}{d\lambda} + 2 g_{\mu\nu}(X)\frac{d\delta X^\mu}{d\lambda}\frac{dX^\nu}{d\lambda} + \left(\partial_\sigma g_{\mu\nu}(X)\right)\delta X^\sigma \frac{dX^\mu}{d\lambda}\frac{dX^\nu}{d\lambda}}\right.$$

$$\left. - \sqrt{g_{\mu\nu}(X)\frac{dX^\mu}{d\lambda}\frac{dX^\nu}{d\lambda}}\right)$$

$$= \int d\lambda\, L \left(\sqrt{1 + \frac{2g_{\mu\nu}(x)}{L^2} \frac{d\delta x^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} + \frac{(\partial_\sigma g_{\mu\nu}(x))}{L^2} \delta x^\sigma \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} - 1 \right)$$

$$\cong \int d\lambda \frac{L}{2L^2} \Big(\underbrace{2g_{\mu\nu}(x) \frac{d\delta x^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} + (\partial_\sigma g_{\mu\nu}(x)) \delta x^\sigma \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \Big)$$

$$= \frac{d}{d\lambda}\left(2g_{\mu\nu} \frac{dx^\nu}{d\lambda} \delta x^\mu\right) - \frac{d}{d\lambda}\left(\frac{2g_{\mu\nu}}{L} \frac{dx^\nu}{d\lambda}\right) \delta x^\mu$$

$$\Longrightarrow \quad L \frac{d}{d\lambda}\left(\frac{2g_{\mu\sigma}}{L} \frac{dx^\mu}{d\lambda}\right) - \partial_\sigma g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0$$

use $d\lambda = dl \Rightarrow L = 1$

$$\Longrightarrow \quad \boxed{\frac{d}{dl}\left(g_{\mu\sigma} \frac{dx^\mu}{dl}\right) - \frac{1}{2} (\partial_\sigma g_{\mu\nu}) \frac{dx^\mu}{dl} \frac{dx^\nu}{dl} = 0}$$

<u>More directly:</u>

$$\int d\lambda \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} \equiv \int d\lambda\, L$$

$$\frac{d}{d\lambda}\left(\frac{\delta L}{\delta \frac{dx^\mu}{d\lambda}} \right) = \frac{\delta L}{\delta x^\mu} \qquad \text{E-L equations}$$

$$\rightarrow \frac{d}{d\lambda}\left(\frac{g_{\mu\nu}\, dx^\nu/d\lambda}{L} \right) = \frac{1}{2L} \partial_\mu g_{\nu\sigma} \frac{dx^\nu}{d\lambda} \frac{dx^\sigma}{d\lambda} \quad \text{OK}$$

Carry out the differentiation

$$\left(0 = g_{\mu\sigma} \frac{d^2 x^\mu}{dl^2} + (\partial_\nu g_{\mu\sigma}) \frac{dx^\nu}{dl} \frac{dx^\mu}{dl} - \frac{1}{2} (\partial_\sigma g_{\mu\nu}) \frac{dx^\mu}{dl} \frac{dx^\nu}{dl}\right) g^{\rho\sigma}$$

$$\Rightarrow \frac{d^2 x^\rho}{dl^2} + \underbrace{\frac{1}{2} g^{\rho\sigma} \left(2 \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right)} \frac{dx^\mu}{dl} \frac{dx^\nu}{dl} = 0$$

convenient to give a name to this combination

$$\boxed{\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right)}$$

Christoffel symbols

(not a tensor!) see later

$$\Downarrow$$

$$\boxed{\frac{d^2 x^\rho}{dl^2} + \Gamma^\rho_{\mu\nu} \frac{dx^\mu}{dl} \frac{dx^\nu}{dl} = 0}$$

Geodesic equation.

Alternative derivation

Go to locally flat coordinates $y^\rho(x)$

$$ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu = \delta_{\mu\nu}\, dy^\mu dy^\nu$$

with $g_{\mu\nu}(x) = \delta_{\rho\sigma} \dfrac{\partial y^\rho}{\partial x^\mu} \dfrac{\partial y^\sigma}{\partial x^\nu}$

A geodesic in locally flat coordinates is a straight line

$$\frac{d^2 y^\rho}{dl^2} = 0 \qquad \left(\text{i.e. } \frac{dy^\rho}{dl} = \text{tangent to the curve} = \text{constant} \right)$$

$$\frac{dy^\rho}{dl} = \frac{\partial y^\rho}{\partial x^\lambda} \frac{dx^\lambda}{dl}$$

$$\frac{\partial x^\sigma}{\partial y^\rho} \left(0 = \frac{d^2 y^\rho}{dl^2} = \frac{d}{dl}\left(\frac{\partial y^\rho}{\partial x^\lambda} \frac{dx^\lambda}{dl} \right) = \frac{\partial y^\rho}{\partial x^\lambda} \frac{d^2 x^\lambda}{dl^2} + \frac{\partial^2 y^\rho}{\partial x^\mu \partial x^\nu} \frac{dx^\mu}{dl} \frac{dx^\nu}{dl} \right)$$

$$\Rightarrow \boxed{\frac{d^2 x^\lambda}{dl^2} + \underbrace{\frac{\partial x^\lambda}{\partial y^\rho} \frac{\partial^2 y^\rho}{\partial x^\mu \partial x^\nu}}_{\text{alternative expression for } \Gamma^\lambda_{\mu\nu}} \frac{dx^\mu}{dl} \frac{dx^\nu}{dl} = 0}$$

<u>Lesson</u>: a straight line in curved coordinates does <u>not</u> look straight!

<u>Important</u>: $\Gamma^\lambda_{\mu\nu}$ is <u>NOT</u> a tensor!

under $y \to x'$

$$\frac{\partial y^\rho}{\partial x'^\nu} = \frac{\partial x^\sigma}{\partial x'^\nu} \frac{\partial y^\rho}{\partial x^\sigma}$$

$$\Gamma'^\lambda_{\mu\nu} = \frac{\partial x'^\lambda}{\partial y^\rho} \frac{\partial^2 y^\rho}{\partial x'^\mu \partial x'^\nu} = \dots =$$

extra piece!

$$= \underbrace{\frac{\partial x'^\lambda}{\partial x^\eta} \frac{\partial x^\omega}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x^\nu} \Gamma^\eta_{\omega\sigma}}_{\text{OK!}} + \boxed{\frac{\partial x'^\lambda}{\partial x^\eta} \frac{\partial^2 x^\eta}{\partial x'^\mu \partial x'^\nu}}$$