Chapter 6

Review

Let us review what we have seen so far before we proceed.

6.1 Electromagnetism

Maxwell's equations:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad \text{(Gauss Law)} \tag{6.1}$$
$$\nabla \cdot \mathbf{B} = 0 \quad \text{(Nonexistence of Magnetic Monopoles)} \tag{6.2}$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{(Faraday induction Law)} \tag{6.3}$$
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad \text{(Ampere Law)} \tag{6.4}$$

naturally imply charge conservation (divergence of Ampere's Law):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0 \tag{6.5}$$

We may define electromagnetic potentials

$$\mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t} \tag{6.6}$$
$$\mathbf{B} = \nabla \times \mathbf{A} \tag{6.7}$$

which under gauge transformations

$$\phi' = \phi - \frac{\partial f}{\partial t} \tag{6.8}$$
$$\mathbf{A}' = \mathbf{A} + \nabla f \tag{6.9}$$

produce the same electromagnetic fields

$$\mathbf{E}' = \mathbf{E} \tag{6.10}$$
$$\mathbf{B}' = \mathbf{B} \tag{6.11}$$

The Lorenz gauge

$$\nabla \cdot \mathbf{A} + \mu_0 \epsilon_0 \frac{\partial \phi}{\partial t} = 0 \quad \text{(Lorenz Gauge)} \tag{6.12}$$

is particularly useful for electromagnetic waves. In fact, inserting the potentials in the Maxwell Eqs. and imposing the Lorenz gauge, we obtain

$$\Box^2 \phi = -\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} + \nabla^2 \phi = -\frac{\rho}{\epsilon_0} \tag{6.13}$$

$$\Box^2 \mathbf{A} = -\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} + \nabla^2 \mathbf{A} = -\mu_0 \mathbf{j} \tag{6.14}$$

i.e., the potentials propagate according to the classical non-homogenous wave equation with constant speed equal to the speed of light $c^2 = 1/\mu_0\epsilon_0$. **Unification: E&M $\leftrightarrow$ Optics**.

Finally, given the E&M fields, corresponding E&M forces $\mathbf{F}$ act on particles as:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{6.15}$$

6.2 Special Relativity

Postulate 1: The laws of physics are the same in all inertial frames.
Postulate 2: The speed of light is the same in all inertial frames.

Postulate 2 follows from postulate 1, since E&M is a set of physical laws.

6.2.1 Coordinates and Metric

Contravariant coordinates

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z) \tag{6.16}$$

Line element ds

$$ds^2 = \eta_{\mu\nu} dx^\nu dx^\mu \tag{6.17}$$

Metric $\eta_{\mu\nu}$

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.18}$$

Covariant coordinates x_μ

$$x_\mu = n_{\mu\nu} x^\nu = (-ct, x, y, z) \tag{6.19}$$

Similarly,

$$x^\mu = \eta^{\mu\nu} x_\nu, \tag{6.20}$$

where $\eta^{\mu\nu}$ inverse metric. Flat space: $\eta^{\mu\nu} = \eta_{\mu\nu}$.
Einstein sum convention: crossed repeated indices are summed over, e.g. $\eta^{\mu\nu} x_\nu \equiv \sum_{\nu=0}^{3} \eta^{\mu\nu} x_\nu$

6.2.2 Invariance of the Line Element:

Under 3d spatial rotations, coordinates transform as

$$x^{\mu\prime} = \frac{\partial x^{\mu\prime}}{\partial x^{\nu}} x^{\nu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \tag{6.21}$$

with

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.22}$$

such that the 3d spatial line element

$$l^2 = x^2 + y^2 + z^2 = (x')^2 + (y')^2 + (z')^2 = l'^2 \tag{6.23}$$

is invariant.

Similary, under a boost with velocity v in the x-direction, the Lorentz transformations with

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.24}$$

where

$$\beta = \frac{v}{c} < 1 \tag{6.25}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} > 1 \tag{6.26}$$

leave the 4-d line element $s^2 = -c^2t^2 + x^2 + y^2 + z^2$ invariant.

6.2.3 Time Dilation and Space Contraction

As a result, we have time dilation:

$$\Delta t' = \Delta t/\gamma \tag{6.27}$$

and space contraction

$$\Delta x' = \gamma \Delta x \tag{6.28}$$

6.2.4 Tensors

Tensors defined according to their Lorentz transformations:

$$T'^{\mu\nu} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}T^{\alpha\beta} \tag{6.29}$$

scalar: tensor of rank 0 (invariant), vector: rank 1, matrix: rank 2, etc...

Example: 4-velocity U^μ:

$$U^\mu = \frac{dx^\mu}{d\tau} = \left(\frac{dx^0}{d\tau}, \frac{dx^i}{d\tau}\right) = \left(\frac{cdt}{d\tau}, \gamma\frac{dx^i}{dt}\right) = (\gamma c, \gamma \mathbf{v}) = \gamma(c, \mathbf{v}) \tag{6.30}$$

4-momentum (massive particles):

$$P^\mu \equiv mU^\mu = (\gamma mc, \gamma m\mathbf{v}) \equiv \left(\frac{E}{c}, \mathbf{p}\right) \qquad \text{Momentum (massive particles)} \tag{6.31}$$

Classical limit ($v \ll c$ we have $\gamma = (1-\beta^2)^{-1/2} \approx 1 + \beta^2/2 + O(\beta^4)$):

$$E = \gamma mc^2 \approx mc^2 + \frac{1}{2}mv^2 + O(\beta^4) \tag{6.32}$$

$$\mathbf{p} = \gamma m\mathbf{v} \approx m\mathbf{v} + O(\beta^3) \tag{6.33}$$

More generally, for massive and massless particles:

$$P^\mu = \frac{dx^\mu}{d\lambda} \equiv \left(\frac{E}{c}, \mathbf{p}\right) \qquad \text{Momentum (massive and massless particles)} \tag{6.34}$$

where λ parametrizes the trajectory. Massive particles: $\lambda = \tau/m$. Massless particles: $\tau = m = 0$, so choose something else or replace $\lambda \to t$. Finally

$$P^\mu P_\mu = -\left(\frac{E}{c}\right)^2 + p^2 = -m^2c^2 \qquad \to \qquad E^2 = (pc)^2 + (mc^2)^2 \tag{6.35}$$

6.2.5 Doppler Effect

Applying the Lorentz transformations to $P^\mu = (E/c, \mathbf{p})$ for a photon, we have

$$E'_\gamma = \sqrt{\frac{1-\beta}{1+\beta}}\, E_\gamma \tag{6.36}$$

and since $E_\gamma = h\nu$:

$$\nu' = \sqrt{\frac{1-\beta}{1+\beta}}\, \nu \tag{6.37}$$

or

$$\lambda' = \sqrt{\frac{1+\beta}{1-\beta}}\, \lambda \qquad \text{Doppler Redshift} \tag{6.38}$$

The redshift z is defined as

$$z = \frac{\Delta\lambda}{\lambda} = \frac{\lambda' - \lambda}{\lambda} = \sqrt{\frac{1+\beta}{1-\beta}} - 1 \approx \sqrt{(1+\beta)^2} - 1 = \frac{v}{c} \tag{6.39}$$

6.2.6 Covariant Formulation

Finally, one can show that the electromagnetic equations can be written in terms of tensors in a covariant form. Defining:

$$
\begin{aligned}
j^\mu &= (c\rho, \mathbf{j}) && (6.40)\\
A^\alpha &= (\phi/c, \mathbf{A}) && (6.41)\\
F^{\mu\nu} &= \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu} && (6.42)\\
f^\mu &= qF^{\mu\nu}U_\nu && (6.43)
\end{aligned}
$$

we have charge conservation:

$$\frac{\partial j^\mu}{\partial x^\mu} = 0\,, \qquad (6.44)$$

Wave equation:

$$\Box A^\alpha = -\mu_0 j^\alpha\,, \qquad (6.45)$$

Gauge transformation:

$$A'^\alpha = A^\alpha + \frac{\partial f}{\partial x_\alpha} \qquad (6.46)$$

Lorenz gauge:

$$\frac{\partial A^\alpha}{\partial x^\alpha} = 0 \qquad (6.47)$$

Maxwell's equations:

$$
\begin{aligned}
\frac{\partial F^{\mu\nu}}{\partial x^\nu} &= \mu_0 j^\mu && (6.48)\\
\frac{\partial F_{\mu\nu}}{\partial x^\sigma} + \frac{\partial F_{\sigma\mu}}{\partial x^\nu} + \frac{\partial F_{\nu\sigma}}{\partial x^\mu} &= 0 && (6.49)
\end{aligned}
$$

and Lorentz force:

$$f^\mu = qF^{\mu\nu}U_\nu \qquad (6.50)$$

6.2.7 Energy-Momentum Tensor

The energy-momentum tensor $T^{\mu\nu}$ is generally defined as

$T^{\mu\nu}$ = "flux of P^μ across surface of constant x^ν" = P^μ per surface $\perp$ to x^ν.

e.g.

T^{00}: density of $P^0 = E$: energy density

T^{ii}: flux of P^i in the x^i direction : force f^i per area $\perp$ to x^i = pressure

For a perfect fluid:

$$T^{\alpha\beta} = (\rho + P)U^\alpha U^\beta + P\eta^{\alpha\beta} \qquad (6.51)$$

6.3 General Relativity

6.3.1 Equivalence Principle

Locally inertial frames: freely-falling frames in small enough regions for which special relativity holds locally.

Weak Equivalence Principle (WEP): "In small enough regions of space-time, the *motion* of freely-falling particles is the same in a uniform gravitational field and in a uniformly accelerated frame, i.e. the *laws of Mechanics* take the same form as in an unaccelerated frame in the absence of gravitation. As a result, at every point of space-time in an arbitrary gravitational field, it is possible to choose a "locally inertial frame" such that in small enough regions the *laws of Mechanics* reduce to those of special relativity."

Strong Equivalence Principle (SEP): Replace *laws of Mechanics* by *laws of Physics* above.

6.3.2 Geodesics

K′ frame: freely-falling coordinates ξ^α,
K frame: coordinates x^β.

$$\frac{d^2\xi^\alpha}{d\tau^2} = 0 \tag{6.52}$$

Change $\xi^\alpha \to x^\beta$:

$$\frac{d^2x^\gamma}{d\tau^2} + \Gamma^\gamma_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \tag{6.53}$$

where the *affine connection* $\Gamma^\gamma_{\mu\nu}$

$$\Gamma^\gamma_{\mu\nu} = \frac{\partial x^\gamma}{\partial \xi^\beta}\frac{\partial^2\xi^\beta}{\partial x^\mu \partial x^\nu} \tag{6.54}$$

Similarly, the *metric tensor* $g_{\mu\nu}$ in coordinates x^μ:

$$g_{\mu\nu} = \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial \xi^\beta}{\partial x^\nu}\eta_{\alpha\beta} \tag{6.55}$$

6.3.3 Metric and Connection

Differentiating Eq. 1.163, changing indices and adding:

$$\Gamma^\sigma_{\mu\lambda} = \frac{g^{\sigma\nu}}{2}\left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda} - \frac{\partial g_{\lambda\mu}}{\partial x^\nu} + \frac{\partial g_{\nu\lambda}}{\partial x^\mu}\right) \tag{6.56}$$

One can show that in the Newtonian limit with

$$g_{\alpha\beta} \quad = \quad \eta_{\alpha\beta} + h_{\alpha\beta}(x), \qquad \text{with } h_{\alpha\beta}(x) \ll \eta_{\alpha\beta} \tag{6.57}$$

the geodesics equation gives

$$\frac{d^2\mathbf{x}}{dt^2} = \frac{c^2}{2}\nabla h_{00} = -\nabla\Psi \tag{6.58}$$

and with appropriate boundary conditions

$$g_{00} = -(1+2\Psi) \tag{6.59}$$

6.3.4 Time Dilation and Gravitational Redshift

Therefore, the ratio of times between 1 and 2 is

$$\frac{dt_2}{dt_1} = \left(\frac{g_{00}(x_2)}{g_{00}(x_1)}\right)^{-1/2} \tag{6.60}$$

i.e. the ratio of frequencies $\nu \propto 1/dt$ will be

$$\frac{\nu_2}{\nu_1} = \left(\frac{g_{00}(x_2)}{g_{00}(x_1)}\right)^{1/2} \tag{6.61}$$

Weak field regime: $g_{00} = -(1+2\Psi)$ and

$$\frac{\delta\nu}{\nu_1} = \frac{\nu_2 - \nu_1}{\nu_1} \approx \Psi(x_2) - \Psi(x_1) \tag{6.62}$$

6.3.5 General Covariance

Equivalence Principle: Gravitational effects can be obtained by writing equations for general gravitational fields in a locally inertial frame where gravitational effects disappear (e.g. $d\xi^2/d\tau^2 = 0$) and transforming to the Laboratory coordinates to find the equation in the Lab. frame.

Principle of General Covariance: alternative to the Equivalence Principle (same physical content).

Principle of General Covariance: A physical equation holds in general gravitational fields (i.e. in general relativity) if:

a) the equation holds in the absence of gravitation; i.e. it agrees with special relativity when $g_{\mu\nu} = \eta_{\mu\nu}$ and $\Gamma^{\alpha}{}_{\mu\nu} = 0$.

b) the equation is *generally* covariant, i.e. it preserves its form under a *general* coordinate transformation.

Volume Element

Define the determinant of the metric:

$$g = \mathrm{Det}\ g_{\mu\nu} \tag{6.63}$$

from which we can show that

$$\sqrt{-g'}\ d^4x' = \sqrt{-g}\ d^4x \tag{6.64}$$

i.e. $\sqrt{-g}\ d^4x$ is an *invariant* (scalar) volume element.

6.3.6 Transformation of the Affine Connection

The affine connection was defined as

$$\Gamma^{\lambda}_{\mu\nu} = \frac{\partial x^{\lambda}}{\partial \xi^{\alpha}} \frac{\partial^2 \xi^{\alpha}}{\partial x^{\mu} \partial x^{\nu}} \tag{6.65}$$

and is not a tensor as it transforms as

$$\Gamma'^{\lambda}_{\mu\nu} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\tau}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \Gamma^{\rho}_{\tau\sigma} - \frac{\partial x^{\rho}}{\partial x'^{\nu}} \frac{\partial x^{\sigma}}{\partial x'^{\mu}} \frac{\partial^2 x'^{\lambda}}{\partial x^{\rho} \partial x^{\sigma}} \tag{6.66}$$

6.3.7 Covariant Differentiation

For a contravariant vector:

$$V'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} V^{\nu}, \tag{6.67}$$

and its derivative is

$$\frac{\partial V'^{\mu}}{\partial x'^{\lambda}} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} \frac{\partial V^{\nu}}{\partial x^{\rho}} + \frac{\partial^2 x'^{\mu}}{\partial x^{\nu} \partial x^{\rho}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} V^{\nu}. \tag{6.68}$$

Combining the transformations for $\Gamma^{\lambda}_{\mu\nu}$ and V^{ν} we have

$$\Gamma'^{\mu}_{\lambda\kappa} V'^{\kappa} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} \Gamma^{\nu}_{\rho\sigma} V^{\sigma} - \underbrace{\frac{\partial^2 x'^{\mu}}{\partial x^{\rho} \partial x^{\sigma}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} V^{\sigma}}_{\frac{\partial^2 x'^{\mu}}{\partial x^{\rho} \partial x^{\nu}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} V^{\nu}} \tag{6.69}$$

Adding the two equations above, the inhomogeneous terms cancel out and we get

$$\frac{\partial V'^{\mu}}{\partial x'^{\lambda}} + \Gamma'^{\mu}_{\lambda\kappa} V'^{\kappa} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} \left(\frac{\partial V^{\nu}}{\partial x^{\rho}} + \Gamma^{\nu}_{\rho\sigma} V^{\sigma} \right) \tag{6.70}$$

The combination in brackets is the *covariant derivative*, which transforms as a *tensor*:

$$\nabla_{\lambda} V^{\mu} = V^{\mu}{}_{;\lambda} = \frac{\partial V}{\partial x^{\lambda}} + \Gamma^{\mu}_{\lambda\kappa} V^{\kappa} \tag{6.71}$$

Extended to a general tensor:

$$T^{\mu\sigma}{}_{\lambda;\rho} = \frac{\partial T^{\mu\sigma}{}_{\lambda}}{\partial x^{\rho}} + \Gamma^{\mu}_{\rho\nu} T^{\nu\sigma}{}_{\lambda} + \Gamma^{\sigma}_{\rho\nu} T^{\mu\nu}{}_{\lambda} - \Gamma^{\kappa}_{\lambda\rho} T^{\mu\sigma}{}_{\kappa} \tag{6.72}$$

The covariant derivative of the metric is zero, as can be checked, using Eq. 1.172:

$$g_{\mu\nu;\lambda} = \frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} - \Gamma^{\rho}_{\lambda\mu} g_{\rho\nu} - \Gamma^{\rho}_{\lambda\nu} g_{\mu\rho} = 0 \tag{6.73}$$

Importance of covariant derivatives for forming covariant equations:
1) They transform tensors into tensors, i.e. if $A^{\mu\nu}$ is a tensor, so is $\nabla_{\lambda} A^{\mu\nu}$.
2) They reduce to ordinary derivatives in the absence of gravity (when $g_{\mu\nu} = \eta_{\mu\nu}$ and $\Gamma^{\lambda}_{\mu\nu} = 0$).

Therefore, the principle of general covariance allows us to apply the following algorithm to obtain equations that are generally covariant and true in the presence of gravity:
a) Write the equation in special relativity (which holds in the absence of gravitation)
b) Replace $\eta_{\mu\nu} \to g_{\mu\nu}$
c) Replace $\partial/\partial x^{\mu} \to \nabla_{\mu}$.

6.4 Curvature

The connection is not a tensor, but the combination defined as the *Riemann curvature tensor*

$$R^{\lambda}{}_{\mu\nu\kappa} = \frac{\partial\Gamma^{\lambda}_{\mu\nu}}{\partial x^{\kappa}} - \frac{\partial\Gamma^{\lambda}_{\mu\kappa}}{\partial x^{\nu}} + \Gamma^{\eta}_{\mu\nu}\Gamma^{\lambda}_{\kappa\eta} - \Gamma^{\eta}_{\mu\kappa}\Gamma^{\lambda}_{\nu\eta} \quad \text{(Riemann Tensor)} \tag{6.74}$$

is indeed a tensor:

$$R'^{\tau}{}_{\rho\sigma\eta} = \frac{\partial x'^{\tau}}{\partial x^{\lambda}}\frac{\partial x^{\mu}}{\partial x'^{\rho}}\frac{\partial x^{\nu}}{\partial x'^{\sigma}}\frac{\partial x^{\kappa}}{\partial x'^{\eta}}R^{\lambda}{}_{\mu\nu\kappa} \tag{6.75}$$

Tensors of lower rank by contracting the Riemann Tensor. Ricci tensor:

$$R_{\mu\nu} = g^{\lambda\kappa}R_{\lambda\mu\kappa\nu} = R^{\kappa}{}_{\mu\kappa\nu} \quad \text{(Ricci Tensor)} \tag{6.76}$$

Ricci scalar:

$$R = g^{\mu\nu}R_{\mu\nu} = R^{\mu}{}_{\mu} \quad \text{(Ricci Scalar)} \tag{6.77}$$

It can also be shown that these are the only tensor and scalar that can be formed from the Riemann tensor and the metric.

6.4.1 Commutation of Covariant Derivatives

Covariant derivative to a covariant vector V_μ twice in reverse order leads to

$$V_{\mu;\nu;\kappa} - V_{\mu;\kappa;\nu} = -R^{\sigma}{}_{\mu\nu\kappa}V_{\sigma} \tag{6.78}$$

Therefore, if the Riemann tensor vanishes, covariant derivatives commute (as they should in flat space). For a space-time with curvature, covariant derivatives do not commute.

One can show a number of properties of the Riemann Tensor, these lead to the Bianchi Identities, which imply:

$$(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R)_{;\mu} = 0 \tag{6.79}$$

6.5 Einstein Equations

Finally, imposing that the gravitational field equations must satisfy certain conditions, such as being tensorial, containing at most 2 derivatives of the metric, being consistent with the Bianchi identities, and reducing to Newtonian gravity in the appropriate limit, one finds that

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \qquad \text{(Einstein Equations)} \tag{6.80}$$

where

$$G_{\mu\nu} = R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R \tag{6.81}$$

This result can also be obtained by the Einstein-Hilbert action:

$$S_{\text{EH,vac}} = \int d^4x\sqrt{-g}\; R\,. \tag{6.82}$$

if we require this action to be stationary under variations with respect to the metric $g^{\mu\nu}$.

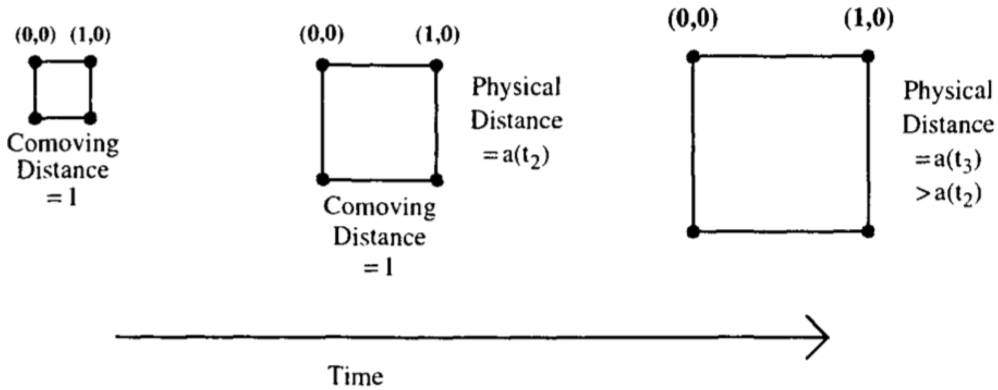

Figure 6.1: Scale factor and expansion. Comoving coordinates do not change, but physical coordianates expand with the scale factor $a(t)$. (Dodelson).

6.6 Expansion of the Universe

Cosmological Principle: Assumption that the Universe is homogeneous (same at every point, therefore symmetric under translations) and isotropic (same in all directions, therefore symmetric under rotations).

Expanding universe: useful to define *comoving coordinates* $\mathbf{x}$: do not change with the expansion, parametrized in terms of the scale factor $a(t)$ (see Fig. 6.1.

Then physical distances r change with change such that

$$\text{physical distance} = a(t) \times \text{ comoving distance.} \tag{6.83}$$

or

$$\mathbf{r}(t) = a(t)\mathbf{x} \tag{6.84}$$

6.7 The Friedmann-Robertson-Walker metric

Generalizes Minkowski metric to include expansion on the spatial hypersurfaces, maintaining spatial isotropy and homogeneity. *Flat* Universe it is given by

$$ds^2 = -dt^2 + a^2(t)dl^2 \tag{6.85}$$

where

$$dl^2 = dx^2 + dy^2 + dz^2 = dD^2 + D^2 d\alpha^2 \tag{6.86}$$

and

$$d\alpha^2 = d\theta^2 + \sin^2\theta d\phi^2 \tag{6.87}$$

For universes with curvature k, generalize

$$dl^2 = R^2\left[dD^2 + f_k^2(D)d\alpha^2\right] \tag{6.88}$$

$$= R^2\left[\frac{dD_A^2}{1-kD_A^2} + D_A^2 d\alpha^2\right] \quad \text{(3d curved space)} \tag{6.89}$$

such that:

$$D_A = f_k(D) = \frac{\sin(\sqrt{k}D)}{\sqrt{k}} = \begin{cases} \sinh(D), & k=-1, \quad \text{Negative Curvature,} \quad \text{Open Universe} \\ D, & k=0, \quad \text{Zero Curvature,} \quad \text{Flat Universe} \\ \sin(D), & k=+1, \quad \text{Positive Curvature,} \quad \text{Closed Universe} \end{cases} \tag{6.90}$$

6.8 The Friedmann Equations

(FRW metric + Einstein Equations) → Friedmann Equations:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho \tag{6.91}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) \tag{6.92}$$

with curvature, generalizes to

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \tag{6.93}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) \tag{6.94}$$

In a universe with no curvature, the density is called critical

$$\rho_{\text{crit}}(t) = \frac{3H^2(t)}{8\pi G} \tag{6.95}$$

Define the density parameter

$$\Omega_i(t) = \frac{\rho_i(t)}{\rho_{\text{crit}}(t)} \tag{6.96}$$

and the Friedmann equation becomes

$$E^2(t) = \frac{H^2(t)}{H_0^2} = \left[\Omega_k\ a^{-2} + \Omega_{\text{m}}\ a^{-3} + \Omega_{\text{r}}\ a^{-4} + \Omega_\Lambda\right] \tag{6.97}$$

where

$$\Omega_k = -k/H_0^2 = 1 - (\Omega_{\text{m}} + \Omega_{\text{r}} + \Omega_\Lambda) \tag{6.98}$$

For a Universe with both matter and cosmological constant, we have

$$a(t) = \left(\frac{\Omega_{\text{m}}}{\Omega_\Lambda}\right)^{1/3} \sinh^{2/3}\left(\frac{3\sqrt{\Omega_\Lambda}H_0}{2}t\right) \quad \text{(Matter + Cosmological Constant)} \tag{6.99}$$

In the context of an expanding universe, the gravitational (dynamical) redshift is due to the stretch of space-time itself and relates to the scale factor

$$1 + z = \frac{1}{a} \tag{6.100}$$

6.9 Cosmological Distances

6.9.1 Comoving Radial Distance

The comoving radial distance D can be obtained by considering the a radial path of photons, in which we have $d\alpha^2 = 0$ (radial) and $ds^2 = -dt^2 + a^2(t)dD^2 = 0$ (photons), so that D can be expressed as

$$\begin{aligned} D &= \int dD = \int_t^{\text{age}} \frac{dt}{a(t)} = \int_a^1 \frac{da}{\dot{a}a} = -\int_z^0 \frac{dz}{H(z)} \\ &= \int_0^z \frac{dz}{H(z)} \end{aligned} \tag{6.100}$$

where we used $da = -a^2 dz$ and $H(z) = \dot{a}/a$. Notice that D depends on the curvature only via the Hubble parameter from the Friedmann's equations. We may also define a *physical* radial distance $d_p = a(t)D$.

6.9.2 Comoving Horizon

The comoving horizon D_H is similar to D, but instead of integrating from $z = 0$ to a certain redshift z, we integrate from z to $z = \infty$, effectivelly finding the comoving size of the universe at z:

$$D_H = \int_0^t \frac{dt}{a(t)} = \int_0^a \frac{da}{\dot{a}a} = \int_z^\infty \frac{dz}{H(z)} \tag{6.101}$$

We may also define a *physical* horizon $d_H = a(t)D_H$.

6.9.3 Angular Diameter Distance

The comoving angular diameter distance D_A is defined such that it gives an object's comoving size dl when it is multiplied by the object angular size $d\alpha$

$$dl = D_A d\alpha \tag{6.102}$$

From the metric definition, with $dD = 0$ we can see that it is given in terms of D by

$$D_A = f_k(D) = \frac{\sin(\sqrt{k}D)}{\sqrt{k}} = \begin{cases} \sinh(D), & k = -1, \quad \text{Negative Curvature, Open Universe} \\ D, & k = 0, \quad \text{Zero Curvature, Flat Universe} \\ \sin(D), & k = +1, \quad \text{Positive Curvature, Closed Universe} \end{cases} \tag{6.103}$$

or similarly, with $k = -H_0^2\Omega_k$:

$$D_A = f_k(D) = \frac{\sin[\sqrt{-\Omega_k}H_0 D]}{\sqrt{-\Omega_k}H_0} = \begin{cases} \frac{\sinh[\sqrt{\Omega_k}H_0 D]}{\sqrt{\Omega_k}H_0 D}, & \Omega_k > 0, \quad \text{Negative Curvature, Open Universe} \\ D, & \Omega_k = 0, \quad \text{Zero Curvature, Flat Universe} \\ \frac{\sin[\sqrt{-\Omega_k}H_0 D]}{\sqrt{-\Omega_k}H_0 D}, & \Omega_k < 0, \quad \text{Positive Curvature, Closed Universe} \end{cases} \tag{6.104}$$

6.9.4 Luminosity Distance

The *physical* luminosity distance d_L is defined such that the Euclidean relation remains valid for the comoving flux, i.e.

$$F = \frac{L}{4\pi d_L^2} \tag{6.105}$$

and comparing with the previous equation, we conclude that

$$d_L = \frac{D_A}{a} = \frac{d_A}{a^2} \tag{6.106}$$

In the case of a flat universe we have

$$d_L = \frac{D}{a} = \frac{d}{a^2} \qquad \text{(Flat)} \tag{6.107}$$

In any case, the relation $a^2 d_L = d_A$ is always true for FRW cosmologies, independent of curvature and/or cosmology. It provides a consistency check for the homogeneity and isotropy of the Universe.

Finally, the *comoving* luminosity distance is

$$D_L = \frac{d_L}{a} = \frac{D_A}{a^2} = \frac{f_k(D)}{a^2} \tag{6.108}$$

6.9.5 Comoving Volume

The comoving volume element in spherical coordinates is given by

$$dV(z) = (D_A d\theta)(D_A \sin\theta d\phi)dD = \frac{D_A^2(z)}{H(z)} dz d\Omega, \tag{6.108}$$

6.9.6 Comoving versus Physical

physical and comoving version. The physical distance d is always obtained by multiplying the comoving distance D by the scale factor $a(t)$. This holds also for the luminosity and angular-diameter distances such that:

$$\begin{aligned} d_p &= a(t)D && (6.109)\\ d_H &= a(t)D_H && (6.110)\\ d_A &= a(t)D_A && (6.111)\\ d_L &= a(t)D_L && (6.112) \end{aligned}$$

and the physical volume is

$$dV_{\text{phys}} = (d_A d\theta)(d_A \sin\theta d\phi)d(d_p) = a^3(t)\frac{D_A^2(z)}{H(z)} dz d\Omega = a^3(t)dV \tag{6.113}$$

6.10 Energy Evolution

The Bianchi identity says that the covariant derivative of the Einstein Tensor is zero:

$$\nabla_\mu G^{\mu\nu} = 0 \tag{6.114}$$

which, through the Einstein equations, automatically imply that the Energy-Momentum tensor is covariantly conseved:

$$\nabla_\mu T^{\mu\nu} = 0 \tag{6.115}$$

the $\nu = 0$ equation implies ($T^{00} = g^{00}T^0{}_0 = \rho$ and $T^{ij} = g^{ik}T^i{}_k = -\delta_{ik}/a^2(-\delta_{ik}P) = \delta_{ij}P/a^2$):

$$\begin{aligned}
\nabla_\mu T^{\mu 0} &= \partial_\mu T^{\mu 0} + \Gamma^\mu_{\mu\lambda}T^{\lambda 0} + \Gamma^0_{\mu\lambda}T^{\mu\lambda} \\
&= \partial_0 T^{00} + \Gamma^0_{0\lambda}T^{\lambda 0} + \Gamma^i_{i\lambda}T^{\lambda 0} + \Gamma^0_{0\lambda}T^{0\lambda} + \Gamma^0_{i\lambda}T^{i\lambda} \\
&= \partial_0 T^{00} + \Gamma^i_{i\lambda}T^{\lambda 0} + \Gamma^0_{i\lambda}T^{i\lambda} \\
&= \partial_0 T^{00} + \Gamma^i_{i0}T^{00} + \Gamma^0_{ij}T^{ij} \\
&= \partial_0\rho + \delta_{ii}\frac{\dot a}{a}\rho + (\delta_{ij}a\dot a)\left(\frac{\delta_{ij}P}{a^2}\right) \\
&= \frac{\partial\rho}{\partial t} + 3\frac{\dot a}{a}\rho + 3\frac{\dot a}{a}P = 0 \qquad (6.111)
\end{aligned}$$

or with $P = w\rho$:

$$\boxed{\frac{\partial\rho}{\partial t} + 3H(1+w)\rho = 0} \tag{6.112}$$

the general solution to this equation as

$$\begin{aligned}
\frac{d\rho}{dt} &= -3\frac{da/dt}{a}\rho[1+w(t)] \\
\frac{d\rho}{\rho} &= -3[1+w(t)]\frac{da}{a} \\
d\ln\rho &= -3[1+w(t)]d\ln a \\
\ln\rho &= -3\int[1+w(a)]d\ln a + \text{const.} \\
\rho(a) &= \rho(1)\exp\left[-3\int_1^a\frac{(1+w(a))}{a}da\right] \qquad (6.109)
\end{aligned}$$

In terms of redshift z, $a = (1+z)^{-1}$, $da = -(1+z)^{-2}dz$, so that $da/a = -dz/(1+z)$ and:

$$\rho(z) = \rho(0)\exp\left[3\int_0^z\frac{[1+w(z)]}{1+z}dz\right] \tag{6.110}$$

Solutions for constant w

We can find solutions for cases when the universe content is dominated by different species with constant w:

$$\rho(z) = \rho(0)\exp\left[3(1+w)\int_0^z \frac{dz}{1+z}\right] = \rho(0)\exp\left[3(1+w)\ln(1+z)\right] \tag{6.111}$$

or

$$\rho(z) = \rho(0)(1+z)^{3(1+w)} \tag{6.112}$$

6.11 Equilibrium Thermodynamics

distribution function $f(\mathbf{x},\mathbf{p},t)$ of a species in phase space $(\mathbf{x},\mathbf{p})$ and time t, defined such that

$$N = f(\mathbf{x},\mathbf{p},t)d^3xd^3p \tag{6.113}$$

is the number of particles in phase space element d^3xd^3p.

In thermodynamical equilibrium, the distribution function is independent of position angular direction, and given by

$$f(\mathbf{x},\mathbf{p},t) = f(p,t) = \frac{1}{e^{(E-\mu)/T}\pm 1}\begin{cases} + & \text{Fermi-Dirac} \\ - & \text{Bose-Einstein}\end{cases} \tag{6.114}$$

where $E=\sqrt{p^2+m^2}$, and both cases reduce to the Maxwell-Boltzmann distribution in the classical limit (high temperatures and low densitites):

$$f(p,t) \propto e^{-(E-\mu)/T} \qquad \text{Classical} \tag{6.115}$$

number density, energy density and pressure, respectively:

$$n(\mathbf{x},t) = g\int \frac{d^3p}{(2\pi)^3} f(\mathbf{x},\mathbf{p},t) \tag{6.116}$$

$$\rho(\mathbf{x},t) = g\int \frac{d^3p}{(2\pi)^3} E f(\mathbf{x},\mathbf{p},t) \tag{6.117}$$

$$P(\mathbf{x},t) = g\int \frac{d^3p}{(2\pi)^3}\frac{p^2}{3E} f(\mathbf{x},\mathbf{p},t) \tag{6.118}$$

The Boltzmann equation then implies

$$T \propto \frac{1}{a} \tag{6.119}$$

6.12 Boltzmann Equations

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{dx_i}{dt}\frac{\partial f}{\partial x_i} + \frac{dp}{dt}\frac{\partial f}{\partial p} + \frac{d\hat{p}_i}{dt}\frac{\partial f}{\partial \hat{p}_i} = \left(\frac{\partial f}{\partial t}\right)_C \tag{6.120}$$

In equilibrium, the distribution $f(\mathbf{x}, \mathbf{p}, t) = f_0(p, t)$ is either the BE or FD distribution, and the collision term is zero (collisions/reactions in one direction cancelled by terms in opposite direction) such that the collisionless Boltzmann is satisfied and

$$\frac{df_0}{dt} = \frac{\partial f_0}{\partial t} + \frac{dx^i}{dt}\underbrace{\frac{\partial f_0}{\partial x^i}}_{0} + \frac{dp}{dt}\frac{\partial f_0}{\partial p} + \frac{d\hat{p}^i}{dt}\underbrace{\frac{\partial f}{\partial \hat{p}^i}}_{0} = 0 \tag{6.121}$$

$$\rightarrow \frac{\partial f_0}{\partial t} + \frac{dp}{dt}\frac{\partial f_0}{\partial p} = 0 \tag{6.122}$$

For photons

$$P^2 = g_{\mu\nu}P^\mu P^\nu = 0 \rightarrow P^0 = p \tag{6.123}$$

and the Geodesics equation gives

$$\frac{dp}{dt} = -Hp \tag{6.124}$$

For matter

$$P^2 = g_{\mu\nu}P^\mu P^\nu = -m^2 \rightarrow E^2 = p^2 + m^2 \tag{6.125}$$

and the Boltzmann equation leads to

$$\rho \propto \frac{1}{a^3} \tag{6.126}$$

6.13 Thermal History

The Boltzmann equation may be written as

$$\frac{\partial f}{\partial t} - Hp\frac{\partial f}{\partial p} = \frac{1}{E}\left(\frac{\partial f}{\partial t}\right)_C \tag{6.127}$$

or, similarly, integrating over momentum

$$a^{-3}\frac{d(na^3)}{dt} = g\int \frac{d^3p}{(2\pi)^3}\frac{1}{E}\left(\frac{\partial f}{\partial t}\right)_C \tag{6.128}$$

For a general process

$$1 + 2 \leftrightarrow 3 + 4 \tag{6.129}$$

we may evaluate the collision term and obtain for particle 1

$$a^{-3}\frac{d(n_1a^3)}{dt} = n_1^{(0)}n_2^{(0)}\langle\sigma v\rangle\left[\frac{n_3n_4}{n_3^{(0)}n_4^{(0)}} - \frac{n_1n_2}{n_1^{(0)}n_2^{(0)}}\right] \tag{6.130}$$

In chemical equilibrium the collision term is zero and we have the Saha equation

$$\frac{n_3n_4}{n_3^{(0)}n_4^{(0)}} = \frac{n_1n_2}{n_1^{(0)}n_2^{(0)}} \tag{6.131}$$

More generally, we must solve the differential equation while only kinetic equilibrium holds.

We used this equation to study a number of processes in the early universe, namely

* Neutrino decoupling
* Freeze out of neutrons,
* Big Bang nucleosynthesis: formation of light element nuclei.
* Recombination of electrons and protons allowing the decoupling of electrons and photons
* Production of relic dark matter particles.

6.14 Linear Perturbations in the Universe

Gravitational dynamics → space-time *perturbations* in the metric and in the energy-momentum tensor components:

$$\delta g_{\mu\nu}(\mathbf{x},t) \quad : \quad \Psi(\mathbf{x},t), \Phi(\mathbf{x},t) \tag{6.132}$$

$$\delta T_{\mu\nu}(\mathbf{x},t) \quad : \quad \delta\rho(\mathbf{x},t), v_i(\mathbf{x},t), \delta P(\mathbf{x},t), \Pi_{ij}(\mathbf{x},t) \tag{6.133}$$

Fourier transform

$$\delta(\mathbf{k},t) = \int d^3x \; e^{-i\mathbf{k}\cdot\mathbf{x}}\delta(\mathbf{x},t) \tag{6.134}$$

and inverse

$$\delta(\mathbf{x},t) = \int \frac{d^3k}{(2\pi)^3} \; e^{i\mathbf{k}\cdot\mathbf{x}}\delta(\mathbf{k},t) \tag{6.135}$$

lead to

$$\delta(\mathbf{x}) \quad \to \quad \delta(\mathbf{k}) \tag{6.136}$$

$$\frac{\partial}{\partial x_i}\delta(\mathbf{x}) \quad \to \quad ik_i\delta(\mathbf{k}) \tag{6.137}$$

$$\nabla^2\delta(\mathbf{x}) \quad \to \quad -k^2\delta(\mathbf{k}) \tag{6.138}$$

$$\int d^3x'\delta(x')W(x-x') \quad \to \quad \delta(\mathbf{k})W(\mathbf{k}) \tag{6.139}$$

Metric perturbations (Conformal Newtonian Gauge):

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -(1+2\Psi)dt^2 + a^2(1+2\Phi)dx^2 \tag{6.140}$$

$\Psi(x,t)$: Newtonian potential (time-time metric perturbation)
$\Phi(x,t)$ curvature potential (space-space metric perturbation).

This metric leads, in Fourier space to the connection symbols:

$$\Gamma^0_{00} = \dot{\Psi}\,, \qquad \Gamma^0_{0i} = \Gamma^0_{i0} = ik_i\Psi\,, \qquad \Gamma^0_{ij} = \delta_{ij}a^2\left[H + 2H(\Phi-\Psi) + \dot{\Phi}\right] \tag{6.141}$$

$$\Gamma^i_{00} = \frac{ik_i}{a^2}\Psi\,, \qquad \Gamma^i_{0j} = \Gamma^i_{j0} = \delta_{ij}\left(H + \dot{\Phi}\right)\,, \qquad \Gamma^i_{jk} = i\Phi(\delta_{ki}k_j - \delta_{jk}k_i + \delta_{ij}k_k) \tag{6.142}$$

Ricci tensor:

$$R_{00} = -3\frac{\ddot{a}}{a} - \frac{k^2}{a^2}\Psi + 3H(\dot{\Psi} - 2\dot{\Phi}) - 3\ddot{\Phi} \tag{6.143}$$

$$R_{0i} = -2ik_i(\dot{\Phi} - H\Psi) \tag{6.144}$$

$$R_{ij} = \delta_{ij}\left[(a\ddot{a} + 2a^2H^2)[1 + 2(\Phi - \Psi)] + a^2H\left(6\dot{\Phi} - \dot{\Psi}\right) + a^2\ddot{\Phi} + k^2\Phi\right] + k_ik_j(\Phi + \Psi) \tag{6.145}$$

Ricci scalar:

$$R = 6\left(\frac{\ddot{a}}{a} + H^2\right) + \frac{2k^2}{a^2}(\Psi + 2\Phi) - 6H(\dot{\Psi} - 4\dot{\Phi}) + 6\ddot{\Phi} - 12\Psi\left(\frac{\ddot{a}}{a} + H^2\right) \tag{6.146}$$

6.15 Perturbed Boltzmann Equations

FRW metric with perturbations in the Newtonian gauge → Bolzmann equation:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a}\frac{p}{E}\frac{\partial f}{\partial x^i} - \frac{\partial f}{\partial E}\left[\frac{p^2}{E}\dot{\Phi} + \frac{\partial \Psi}{\partial x^i}\frac{p\hat{p}^i}{a} + \frac{p^2}{E}H\right] \quad = \quad \left(\frac{\partial f}{\partial t}\right)_C\,. \tag{6.146}$$

6.15.1 Photons

For photons, $E = p$ and

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i} - p\frac{\partial f}{\partial p}\left[H + \dot{\Phi} + \frac{\partial \Psi}{\partial x^i}\frac{\hat{p}^i}{a}\right] \tag{6.147}$$

Perturbation in the distribution function around equilibrium Planck distribution $f^0(p,t)$:

$$f(\mathbf{x},\mathbf{p},t) = f^0(p,t) + \delta f(\mathbf{x},\mathbf{p},t) \tag{6.148}$$

or similarly in terms of perturbations in the temperature field

$$T(\mathbf{x},\hat{p},t) = T(t) + \delta T(\mathbf{x},\hat{p},t) = T(t)\left[1 + \Theta(\mathbf{x},\hat{p},t)\right] \tag{6.149}$$

where $\Theta(\mathbf{x},\hat{p},t) = \delta T(\mathbf{x},\hat{p},t)/T(t)$, so that

$$\begin{aligned} f(\mathbf{x},\mathbf{p},t) &= \left\{\exp\left[\frac{p}{T(\mathbf{x},\hat{p},t)}\right] - 1\right\}^{-1} \\ &= f^0(p,t) - p\frac{\partial f^0}{\partial p}\Theta \end{aligned} \tag{6.149}$$

or

$$\delta f(\mathbf{x},\mathbf{p},t) = -p\frac{\partial f^0}{\partial p}\Theta \tag{6.150}$$

Keeping only first-order terms, we have

$$\left.\frac{df}{dt}\right|_{\text{1st order}} = -p\frac{\partial f^0}{\partial p}\left[\dot{\Theta} + \frac{\hat{p}_i}{a}\frac{\partial \Theta}{\partial x^i} + \dot{\Phi} + \frac{\hat{p}_i}{a}\frac{\partial \Psi}{\partial x^i}\right] \tag{6.151}$$

Compton scattering of photon off electrons is the main interaction:

$$e^-(\mathbf{q}) + \gamma(\mathbf{p}) \leftrightarrow e^-(\mathbf{q}') + \gamma(\mathbf{p}') \tag{6.152}$$

with amplitude

$$|\mathcal{M}|^2 \approx 8\pi\sigma_T m_e^2 \tag{6.153}$$

and the collision term is given by

$$\left(\frac{\partial f(\mathbf{p})}{\partial t}\right)_C = -p\frac{\partial f^0}{\partial p} n_e \sigma_T \; [\Theta_0 - \Theta + \hat{p}\cdot\mathbf{v}_b] \tag{6.154}$$

where

$$n_e = \int \frac{d^3q}{(2\pi)^3} f_e(\mathbf{q}) \tag{6.155}$$

$$n_e \mathbf{v}_b = \int \frac{d^3q}{(2\pi)^3} f_e(\mathbf{q}) \frac{\mathbf{q}}{m_e} \tag{6.156}$$

$$\Theta_l = \frac{1}{(-i)^l}\int_{-1}^{1}\frac{d\mu}{2}P_l(\mu)\;\Theta \tag{6.157}$$

The full equation becomes

$$\dot{\Theta} + \frac{\hat{p}_i}{a}\frac{\partial \Theta}{\partial x^i} + \dot{\Phi} + \frac{\hat{p}_i}{a}\frac{\partial \Psi}{\partial x^i} = n_e\sigma_T \; [\Theta_0 - \Theta + \hat{p}\cdot\mathbf{v}_b] \tag{6.158}$$

Then,

* Change $t \to \eta$,

* Change to Fourier space,

* use $\mu = \cos(\theta) = \frac{\mathbf{k}\cdot\hat{p}}{k} = \frac{k_i \hat{p}_i}{k} \qquad \to \qquad \hat{p}_i k_i = \mu k$
* Define optical depth: $\tau' \equiv \frac{d\tau}{d\eta} = -n_e\sigma_T a$

and finally

$$\Theta' + ik\mu\Theta + \Phi' + ik\mu\Psi' = -\tau'\left[\Theta_0 - \Theta + \mu v_b\right] \tag{6.159}$$

6.15.2 Dark matter

For cold dark matter it is easier to simply use energy-momentum conservation. But following the Boltzmann equations we also obtain

$$\delta_c' + ik_i v_c^i + 3\Phi' = 0 \tag{6.160}$$
$$(v_c^i)' + \left(\frac{a'}{a}\right) v_c^i + ik_i\Psi = 0 \tag{6.161}$$

or in terms of $\theta_c(\mathbf{x}, t) = \nabla \cdot \mathbf{v}_c(\mathbf{x}, t) = ikv_c$

$$\delta_c' + \theta_c + 3\Phi' = 0 \tag{6.162}$$
$$\theta_c' + \left(\frac{a'}{a}\right) \theta_c - k^2\Psi = 0 \tag{6.163}$$

The 2 equations may be combined to give

$$\delta_c'' + \left(\frac{a'}{a}\right)\delta' + 3\left[\Phi'' + \left(\frac{a'}{a}\right)\Phi'\right] = -k^2\Psi \tag{6.164}$$

6.15.3 Baryons

For baryons, need to consider the interactions

$$e(q) + p(Q) \rightarrow e(q') + p(Q') \tag{6.165}$$
$$e(q) + \gamma(p) \rightarrow e(q') + \gamma(p') \tag{6.166}$$

to obtain

$$\delta_b' + ikv_b + 3\Phi' = 0 \tag{6.167}$$
$$v_b' + \left(\frac{a'}{a}\right) v_c + ik\Psi = \tau'\frac{4\rho_\gamma}{3\rho_b}\left[3i\Theta_1 + v_b\right] \tag{6.168}$$

6.15.4 Neutrinos

Massless neutrinos: similar to photons, but different temperature T^ν and no collision term. Define $\mathcal{N} = \delta T_\nu / T_\nu$, such that

$$\mathcal{N}' + ik\mu\mathcal{N} + \Phi' + ik\mu\Psi' = 0 \tag{6.169}$$

Massive neutrinos: evolution starts as massless neutrinos while they are relativistic. Transition to transition to that of dark matter once they become non-relativistic.

See Ma & Bertschinger 1995 for a careful description of:
1) linear perturbations for all components above and Einstein Equations in both Conformal Newtonian Gauge and Synchronous Gauge.

2) a technique to solve the equations for photons and neutrinos in terms of a multipole expansion in Legendre polynomials.

6.16 Perturbed Einstein Equations

FRW metric with Newtonian perturbations + Einstein Equations:

$$
\begin{aligned}
-k^2\Phi - 3H\left(\dot{\Phi} - H\Psi\right) &= -4\pi G\delta\rho && (6.170)\\
-k^2(\dot{\Phi} - H\Psi) &= 4\pi G(\rho + P)(ik^i v_i) && (6.171)\\
\ddot{\Phi} - H(\dot{\Psi} - 3\dot{\Phi}) - \left(2\frac{\ddot{a}}{a} + H^2\right)\Psi + \frac{k^2}{3a^2}(\Psi + \Phi) &= -4\pi G\delta P && (6.172)\\
-k^2(\Psi + \Phi) &= 32\pi G\rho\Theta_2 && (6.173)
\end{aligned}
$$

Consider first and third equation in a non-expanding universe and static fields:

$$
\begin{aligned}
-k^2\Phi &= -4\pi G\delta\rho && (6.174)\\
\frac{k^2}{3a^2}(\Psi + \Phi) &= -4\pi G\delta P && (6.175)\\
& && (6.176)
\end{aligned}
$$

so adding the first and 3 times the second we have

$$\nabla^2\Psi = 4\pi G(\delta\rho + 3\delta P) \quad (6.177)$$

In General Relativity, pressure perturbation is also a source to the gravitational potential Ψ .

Finally, we saw initial conditions from the Boltzmann/Einstein equations themselves.

We ended the semester looking at the Inflation model as a solution to a number of problems in the Big Bang scenario (horizon problem, flatness problem, unwanted relics), as well as a means of producing and magnifying quantum perturbations in the early Universe, and its implementation with a slowly-rolling scalar field.