PGF5299: Physical Cosmology II

Problem Set 6

(Due November 05, 2021)

1) Spherically Symmetric Lens: Recall that the reduced deflection angle vector $\hat{\vec{\alpha}} = (\alpha_1, \alpha_2)$ is given by

$$\hat{\vec{\alpha}}(\vec{r}_\perp) = 2\int_{-\infty}^{\infty} \vec{\nabla}_\perp \Phi(r_\parallel, \vec{r}_\perp)\; dr_\parallel \tag{1}$$

For a point lens with mass M, the potential is $\Phi(r_\parallel, \vec{r}_\perp) = GM/(r_\parallel^2 + r_\perp^2)^{1/2}$. Therefore, for a vector $\vec{\xi} = \vec{r}_\perp$ on the lens plane at a distance ξ away from M, the deflection becomes

$$\hat{\vec{\alpha}}(\vec{\xi}) = 4GM\frac{\vec{\xi}}{\xi^2} \tag{2}$$

For a more general lens, we can consider the lens made up of point masses on the lens plane, each one at a distance ξ_i from an origin so that at ξ the reduced deflection is

$$\begin{aligned}
\hat{\vec{\alpha}}(\vec{\xi}) &= \sum_i \alpha_i(\vec{\xi} - \vec{\xi}_i) = 4G\sum_i M_i \frac{\vec{\xi} - \vec{\xi}_i}{|\vec{\xi} - \vec{\xi}_i|^2} = 4G\overbrace{\int d^2\xi' \underbrace{\int dz\;\rho(r)}_{\Sigma(\xi')}}^{dM} \frac{\vec{\xi} - \vec{\xi}'}{|\vec{\xi} - \vec{\xi}'|^2} \\
&= 4G\int d^2\xi' \frac{\Sigma(\xi')(\vec{\xi} - \vec{\xi}')}{|\vec{\xi} - \vec{\xi}'|^2}
\end{aligned} \tag{3}$$

or in terms of $\vec{\theta} = \vec{\xi}/D_L$, the deflection $\vec{\alpha}(\vec{\xi}) = \hat{\vec{\alpha}}(\vec{\xi})D_{LS}/D_S$ (in which $\vec{\theta} = \vec{\beta} + \vec{\alpha}$)

$$\begin{aligned}
\vec{\alpha}(\vec{\theta}) &= \frac{D_{LS}}{D_S}4G\int d^2\theta' \frac{D_L\Sigma(D_L\theta')(\vec{\theta} - \vec{\theta}')}{|\vec{\theta} - \vec{\theta}'|^2} = \frac{1}{\pi}\int d^2\theta' \Sigma(D_L\theta')\frac{4\pi G D_{LS} D_L}{D_S}\frac{(\vec{\theta} - \vec{\theta}')}{|\vec{\theta} - \vec{\theta}'|^2} \\
&= \frac{1}{\pi}\int d^2\theta' \kappa(\vec{\theta}')\frac{(\vec{\theta} - \vec{\theta}')}{|\vec{\theta} - \vec{\theta}'|^2}
\end{aligned} \tag{4}$$

where the convergence $\kappa(\vec{\theta}) = \Sigma(D_L\vec{\theta})/\Sigma_{crit}$ and $\Sigma_{crit}^{-1} = 4\pi G D_L D_{LS}/D_S$.

a) For a spherically symmetric lens, argue that the deflection must point towards the lens center (i.e. $\vec{\alpha} = \alpha(\theta)\vec{\theta}/\theta$) and show that its magnitude is

$$\alpha(\theta) \quad = \quad \frac{M(<\theta)}{\pi D_L^2 \Sigma_{crit}} \frac{1}{\theta} = \frac{M(<\theta)}{\pi \theta^2 D_L^2 \Sigma_{crit}} \theta = \theta \ \bar{\kappa}(<\theta) \tag{5}$$

where $M(<\theta)$ is the mass contained within θ and $\bar{\kappa}(<\theta)$ is an average surface density normalized by the critical surface density (i.e. it is the average convergence).

b) Since $\vec{\alpha} = \alpha(\theta)\vec{\theta}/\theta$ and $\theta = (\theta_1^2 + \theta_2^2)^{1/2}$, compute $\partial\alpha_i/\partial\theta_j$, for $i, j = 1, 2$. Express the results in terms of $\alpha(\theta)$ and $d\alpha/d\theta$.

c) Use the previous relations to compute $\kappa(\theta)$, $\gamma_1(\vec{\theta})$, $\gamma_2(\vec{\theta})$ in terms of $\alpha(\theta)$ and $d\alpha/d\theta$.

d) Compute the shear amplitude $|\gamma|(\theta) = (\gamma_1^2 + \gamma_2^2)^{1/2}$ and the average convergence $\bar{\kappa}(<\theta)$ defined as

$$\bar{\kappa}(<\theta) = \frac{1}{\pi\theta^2} \int 2\pi\theta' \ d\theta' \kappa(\theta') \tag{6}$$

and show that

$$|\gamma|(\theta) = \bar{\kappa}(<\theta) - \kappa(\theta) \tag{7}$$

e) Consider the angle φ of $\vec{\theta}$ with the x_1 axis, so that

$$\theta_1 \quad = \quad \theta \cos(\varphi) \tag{8}$$

$$\theta_2 \quad = \quad \theta \sin(\varphi) \tag{9}$$

Express $\gamma_1(\vec{\theta})$ and $\gamma_2(\vec{\theta})$ computed above in terms of φ, instead of θ_1 and θ_2.

f) Define the tangential shear γ_T and the cross-component $\gamma_\times$ as

$$\gamma_T \quad = \quad -(\cos 2\varphi)\gamma_1 - (\sin 2\varphi)\gamma_2 \tag{10}$$

$$\gamma_\times \quad = \quad (\sin 2\varphi)\gamma_1 - (\cos 2\varphi)\gamma_2 \tag{11}$$

Show that for the spherically symmetric case, $\gamma_T = |\gamma| = \bar{\kappa} - \kappa$ and $\gamma_\times = 0$.

2) NFW Profile: Recall the spherical NFW halo profile is given by:

$$\rho(r|M_{vir}, z) = \frac{\rho_s}{cr/r_{vir}(1 + cr/r_{vir})^2} \tag{12}$$

a) Show analytically that the mass contained within the virial radius of this halo is

$$\begin{aligned} M_{vir} &= \int_0^{r_{vir}} dr\ 4\pi r^2 \rho(r) \ = \ 4\pi\rho_s \frac{r_{vir}^3}{c^3} f^{-1} && (13) \\ \text{where} \quad f(c) &= \left[\ln(1+c) - \frac{c}{1+c}\right]^{-1} && (14) \end{aligned}$$

b) Using $r = |\vec{r}|$ with $\vec{r} = (r_\parallel, \vec{r}_\perp)$ and $\vec{r}_\perp = D_L\vec{\theta}$, compute either analytically or numerically the projected surface density profile

$$\Sigma(\theta) = \int_{-r_{vir}}^{r_{vir}} dr_\parallel\ \rho(r) \tag{15}$$

and plot the convergence $\kappa(\theta) = \Sigma(\theta)/\Sigma_{\rm crit}$ versus θ induced by a halo of mass $M_{vir} = 14.0$ at $z_l = 0.5$ on a source galaxy at $z_s = 2.0$. *Note:* Look at Takada and Jain 2003.

3) Lensing Correlations: Recall the two projected fields:

$$\delta_g^{2D}(\hat{n}) = \int d\chi\ W_g(\chi)\ \delta(\vec{x}) \tag{16}$$

$$\kappa(\hat{n}) = \int d\chi\ W_\kappa(\chi)\ \delta(\vec{x}) \tag{17}$$

for which the angular auto(cross) spectra are given by

$$C_{ab}(\ell) \ = \ \int dz\ \frac{H(z)}{\chi^2(z)}\ W_a(z)W_b(z)\ P\left(k_\ell, z\right) \tag{18}$$

where a, b can be κ or g and for a flat cosmology

$$\begin{aligned} W_g(z) &= b_g(z)\frac{dn_g(z)}{dz} && (19) \\ W_\kappa(z) &= \frac{3}{2}\Omega_m \frac{H_0^2\chi(z)}{H(z)}(1+z)\int_z^\infty dz' \left(1 - \frac{\chi(z)}{\chi(z')}\right)\ \frac{dn_g(z')}{dz'} && (20) \\ \chi(z) &= \int_0^z \frac{dz}{H(z)} && (21) \end{aligned}$$

Suppose a galaxy population for which

$$\frac{dn_g(z)}{dz} = A\ z^\alpha \exp[-(z/z_0)^\beta] \tag{22}$$

$$b_g(z) = 1.0 \tag{23}$$

with $\alpha = \beta = 2$, $z_0 = 1$ and $A = \beta/\{z_0^{\alpha+1}\Gamma\left[(\alpha+1)/\beta\right]\} = 4/\sqrt{\pi}$, so that $\int_0^\infty dz\, dn_g/dz = 1$.

a) Plot $W_g(z)$ and $W_\kappa(z)$.

b) Plot $C_{gg}(\ell)$, $C_{\kappa\kappa}(\ell)$ and $C_{\kappa g}(\ell)$ from $\ell = 1$ to $\ell = 1000$ for a fiducial flat ΛCDM cosmology. Use both linear and non-linear $P(k,z)$ from CAMB (i.e. halofit).

c) Now use only the non-linear $P(k,z)$. Show the effect on these 3 angular spectra of increasing either Ω_m or w by 20%. Plot the relative difference in these spectra with respect to the fiducial cosmology: $\Delta C_{ab}/C_{ab}^{fid}$.