Noncommutativity vs. gravity

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Review

- Reviewing noncommutative classical dynamics

Summary

- Introduction

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- The Kepler potential

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- The Kepler potential
- The harmonic oscillator potential

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- The harmonic oscillator potential
- Noncommutative dynamics and the Schwarzschild solution

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- The Kepler potential
- The harmonic oscillator potential
- Noncommutative dynamics and the Schwarzschild solution
- Small scales vs. long scales

Main references

- S. Benczik, L. N. Chang, D. Minic, N. Okamura, S. Rayyan, T. Takeuchi, Phys. Rev. D **66**, 026003 (2002).

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- S. Benczik, L. N. Chang, D. Minic, N. Okamura, S. Rayyan, T. Takeuchi, Phys. Rev. D **66**, 026003 (2002).
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- M. Chaichian, M. M. Sheikh-Jabbari, and A. Turenau, Phys. Rev. Lett. **86**, 2716 (2001).

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- M. Chaichian, M. M. Sheikh-Jabbari, and A. Turenau, Phys. Rev. Lett. **86**, 2716 (2001).
- F. S. Bemfica and H. O. Girotti, th-IFUFRGS (2009).

Noncommutative classical dynamics

- Lagrangian

$$L = a\, v_j\, \dot{q}^j - h_0(q^j, av_j) + a^2\, \dot{v}^i\, \theta_{ij}\, v^j$$

Noncommutative classical dynamics

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- Hamiltonian

$$H(q,p) = h_0(q,p)$$

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- Hamiltonian

$$H(q,p) = h_0(q,p)$$

- Dirac brackets

$$[q^j, q^k]_{DB} = -2\theta^{lk}$$
$$[q^j, p_k]_{DB} = \delta^j_{\ k}$$
$$[p_j, p_k]_{DB} = 0$$

Construction of the physical space

- Physical space

$$x^j \equiv q^j - \theta^{jl} p_l$$

$$k_j \equiv p_j$$

Construction of the physical space

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$$x^j \equiv q^j - \theta^{jl} p_l$$

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- Hamiltonian

$$H(q,p) = H(x^j + \theta^{jl} k_l, k_j)$$

Kepler potential - I

- Hamiltonian

$$H\left(x^j+\theta^{jl}k_l\,,\,k_j\right)=\frac{k^ik_i}{2M}-\frac{\kappa}{\sqrt{\left(x^i+\theta^{ij}k_j\right)\left(x_i+\theta_{il}k^l\right)}}$$

Kepler potential - I

- Hamiltonian

$$H\left(x^j + \theta^{jl} k_l\,,\, k_j\right) = \frac{k^i k_i}{2M} - \frac{\kappa}{\sqrt{\left(x^i + \theta^{ij} k_j\right)\left(x_i + \theta_{il} k^l\right)}}$$

- Weak noncommutativity approximation

$$H\left(x^j + \theta^{jl} k_l\,,\, k_j\right) = \frac{k^i k_i}{2M} - \frac{\kappa}{r} + \frac{\kappa}{2\,r^3}\,\theta^j\,L_j$$

Kepler potential - I

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- Definitions

$$\theta^{lj} \equiv \frac{1}{2}\epsilon^{ljk}\theta_k\,, \qquad L_j \equiv \epsilon_{jil} x^i k^l$$

Kepler potential - I

- Hamiltonian

$$H\left(x^j + \theta^{jl} k_l\,,\, k_j\right) = \frac{k^i k_i}{2M} - \frac{\kappa}{\sqrt{(x^i + \theta^{ij} k_j)\,(x_i + \theta_{il} k^l)}}$$

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$$\theta^{lj} \equiv \frac{1}{2}\epsilon^{ljk}\theta_k\,, \qquad L_j \equiv \epsilon_{jil} x^i k^l$$

- Conserved quantities

$$H\,, \qquad L^2 \equiv L_j L^j\,, \qquad \theta_j L^j$$

Kepler potential - II

- Choosing

$$\theta^1 = \theta^2 = 0\,, \qquad \theta^3 = \theta \neq 0$$

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- Simplified form of the Hamiltonian

$$H\left(x^j + \theta^{jl} k_l\,,\, k_j\right) = \frac{k^i k_i}{2M} - \frac{\kappa}{r} + \frac{\kappa}{2\,r^3}\,\theta\, L_3$$

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- Constants of motion

$$H\,, \qquad L^2\,, \qquad L_3 = 0$$

Kepler potential - III

- Hamiltonian equation of motion

$$\dot{x}_l = [x_l\, , H]_{PB} = \frac{k_l}{M} - \frac{\kappa}{2r^3}\, \epsilon_{lij}\, x^i\, \theta^j$$

Kepler potential - III

- Hamiltonian equation of motion

$$\dot{x}_l \,=\, [x_l\,, H]_{PB} \,=\, \frac{k_l}{M} \,-\, \frac{\kappa}{2r^3}\,\epsilon_{lij}\,x^i\,\theta^j$$

- The components of $\vec{L}$ in terms of coordinates and velocities

$$L_1 \,=\, M\,x_2\,\dot{x}_3 \,-\, M\,x_3\,\dot{x}_2 \,+\, \kappa\,\frac{M}{2r^3}\,\theta\,x_3\,x_1$$

$$L_2 \,=\, M\,x_3\,\dot{x}_1 \,-\, M\,x_1\,\dot{x}_3 \,+\, \kappa\,\frac{M}{2r^3}\,\theta\,x_3\,x_2$$

$$L_3 \,=\, M\,x_1\,\dot{x}_2 \,-\, M\,x_2\,\dot{x}_1 \,-\, \kappa\,\frac{M}{2r^3}\,\theta\,\left(x_1^2 + x_2^2\right)$$

Kepler potential - IV

- Enforcing the two dimensional motion

$$L^2 = L_3^2 \Longrightarrow L_1^2 + L_2^2 = 0 \Longrightarrow L_1 = 0\,, \quad L_2 = 0\,, \quad \forall t$$

$$x_3 = 0\,, \qquad \forall t \Longrightarrow \dot{x}_3 = 0\,, \qquad \forall t$$

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- The two dimensional Hamiltonian

$$H = \frac{k^a k_a}{2M} + \kappa\,\theta\,\frac{x^a \epsilon_{ab} k^b}{2\rho^3} - \frac{\kappa}{\rho}$$

Kepler potential - V

- The two dimensional Lagrangian

$$L = \frac{M}{2}\,\dot{x}^a\dot{x}_a + \kappa\,\theta\,\frac{M}{2\rho^3}\,\dot{x}^a\epsilon_{ab}x^b + \frac{\kappa}{\rho}$$

Kepler potential - V

- The two dimensional Lagrangian

$$L = \frac{M}{2}\,\dot{x}^a \dot{x}_a + \kappa\,\theta\,\frac{M}{2\rho^3}\,\dot{x}^a \epsilon_{ab} x^b + \frac{\kappa}{\rho}$$

- Equations of motion

$$\frac{dL_3}{dt} = 0$$

$$\frac{d(M\dot{\rho})}{dt} - M\,\rho\,\dot{\varphi}^2 - \kappa\,\theta\,\frac{M}{2\rho^2} + \frac{\kappa}{\rho^2} = 0$$

Kepler potential - VI

- Conserved quantities

$$L_3 = M\,\rho^2\,\dot{\varphi} - \kappa\,\theta\,\frac{M}{2\rho}$$

$$E \equiv \frac{M}{2}\,\dot{\rho}^2 + \frac{L_3^2}{2M\rho^2} + \frac{1}{2}\,\kappa\,\theta\,\frac{L_3}{\rho^3} - \frac{\kappa}{\rho}$$

Kepler potential - VI

- Conserved quantities

$$L_3 = M\,\rho^2\,\dot{\varphi} - \kappa\,\theta\,\frac{M}{2\rho}$$

$$E \equiv \frac{M}{2}\,\dot{\rho}^2 + \frac{L_3^2}{2M\rho^2} + \frac{1}{2}\,\kappa\,\theta\,\frac{L_3}{\rho^3} - \frac{\kappa}{\rho}$$

- Equation for the orbit

$$d\varphi = \left(1 + \kappa\theta\,\frac{M}{2\,L_3\rho}\right)\frac{1}{\rho^2}\frac{d\rho}{\left(\frac{2ME}{L_3^2} - \frac{1}{\rho^2} + \frac{2\kappa M}{L_3^2}\frac{1}{\rho} - \kappa\theta\frac{M}{L_3}\frac{1}{\rho^3}\right)^{\frac{1}{2}}}$$

Kepler potential - VII

- Solution

$$\frac{1}{\rho} = \kappa \frac{M}{L_3^2} \left\{ 1 - e \sin \left[\frac{1}{\left(1 - \frac{\kappa^2 \theta M^2}{L_3^3}\right)} (\varphi - \varphi_0 - \theta F(\rho)) \right] \right\}$$

$$e \equiv \sqrt{1 + \frac{2EL_3^2}{\kappa^2 M}}$$

$$F(\rho) \equiv -\frac{M^2 \kappa}{2\, L_3^3} \frac{\left[4E\left(1 + \frac{EL_3^2}{\kappa^2 M}\right) + \left(1 + 6\frac{EL_3^2}{\kappa^2 M}\right)\frac{\kappa}{\rho}\right]}{\left(1 + 2\frac{EL_3^2}{\kappa^2 M}\right)\left(\frac{2ME}{L_3^2} - \frac{1}{\rho^2} + \frac{2\kappa M}{L_3^2}\frac{1}{\rho}\right)^{\frac{1}{2}}}$$

Kepler potential - VIII

- Precession of the orbits

$$\varphi_-^{(1)} = -\frac{\pi}{2}\left(1 - \kappa^2\theta\frac{M^2}{L_3^3}\right) + \varphi_0 + \theta\, F(\rho_-^{(1)})$$

$$\varphi_-^{(2)} = +\frac{3\pi}{2}\left(1 - \kappa^2\theta\frac{M^2}{L_3^3}\right) + \varphi_0 + \theta\, F(\rho_-^{(2)})$$

$$\left(\varphi_-^{(2)} - \varphi_-^{(1)}\right) - 2\pi = -2\pi\theta\kappa^2\frac{M^2}{L_3^3}$$

Harmonic oscillator potential - I

- Lagrangian

$$L' = \frac{1}{1 + M^2\omega^2\theta^2} \times \left(\frac{1}{2} M \dot{x}^a \dot{x}_a + M^2 \omega^2 \theta \dot{x}^a \epsilon_{ab} x^b - \frac{1}{2} M \omega^2 x^a x_a \right)$$

Harmonic oscillator potential - I

- Lagrangian

$$L' = \frac{1}{1 + M^2\omega^2\theta^2} \times \left(\frac{1}{2}\, M\dot{x}^a\dot{x}_a \,+\, M^2\,\omega^2\,\theta\,\dot{x}^a\epsilon_{ab}x^b \,-\, \frac{1}{2}\,M\omega^2 x^a x_a \right)$$

- Conserved quantities

$$l' = M\rho^2\dot{\varphi} \,-\, M^2\,\omega^2\,\theta\,\rho^2$$

$$E' = \frac{1}{2}\,M\dot{\rho}^2 \,+\, \frac{l'^2}{2M\rho^2} \,+\, \frac{1}{2}M\omega^2\,\left(1 \,+\, M^2\omega^2\theta^2\right)\,\rho^2\,.$$

Harmonic oscillator potential - II

- Equation for the orbit

$$\varphi = \varphi_0 + \frac{1}{2}\arcsin\left[\frac{1-\frac{l'^2}{ME'\rho^2}}{\sqrt{1-\omega^2\left(1+M^2\omega^2\theta^2\right)\frac{l'^2}{E'^2}}}\right]$$

$$-\frac{1}{2}\frac{M\omega\theta}{\sqrt{1+M^2\omega^2\theta^2}}\arcsin\left[\frac{1-\frac{M\omega^2\left(1+M^2\omega^2\theta^2\right)}{E'}\rho^2}{\sqrt{1-\omega^2\left(1+M^2\omega^2\theta^2\right)\frac{l'^2}{E'^2}}}\right]$$

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- No limitations on the values of θ

Harmonic oscillator potential - III

- Turning points - Precession of the orbits

$$
\begin{aligned}
\frac{1}{\rho_{\pm}^{2}} &= \frac{ME'}{l'^{2}} \left\{ 1 \mp \sqrt{1-\omega^{2}\left(1+M^{2}\omega^{2}\theta^{2}\right)\frac{l'^{2}}{E'^{2}}} \right. \\
&\times \left. \sin\left[\frac{2\left(\varphi_{\pm}-\varphi_{0}\right)}{1+\frac{M\omega\theta}{\sqrt{1+M^{2}\omega^{2}\theta^{2}}}}\right] \right\}
\end{aligned}
$$

Noncommutativity and gravity I

- Precession of the orbit from general relativity

$$\left(\varphi_{-}^{(2)} - \varphi_{-}^{(1)}\right) - 2\pi = \frac{6\pi M_{*} G}{c^{2} L}$$

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- Precession of the orbit from general relativity

$$\left(\varphi_-^{(2)} - \varphi_-^{(1)}\right) - 2\pi = \frac{6\pi M_* G}{c^2 L}$$

- Estimate for θ

$$\theta = -\frac{3L_3^3}{c^2 L M^4 M_\star G},$$

Noncommutativity and gravity I

- Precession of the orbit from general relativity

$$\left(\varphi_-^{(2)} - \varphi_-^{(1)}\right) - 2\pi = \frac{6\pi M_* G}{c^2 L}$$

- Estimate for θ

$$\theta = -\frac{3L_3^3}{c^2 L M^4 M_\star G},$$

- Calculation of L_3^3

$$L_3^3 = 8\,\pi^3\,\left(1-e^2\right)^{\frac{3}{2}}\,\frac{M^3\,a^6}{\tau^3} - 6\,\pi^2\,\left(1-e^2\right)\frac{M^4 M_* G\,\theta\,a^3}{\tau^2}$$

Noncommutativity and gravity II

- <u>Calculation of θ - Numerical value for the earth planet</u>

$$\theta = -\left(1 - 18\,\frac{\pi^2 a^2}{c^2\tau^2}\right)^{-1} \frac{24\pi^3 a^5\sqrt{1-e^2}}{c^2 M M_\star G \tau^3}$$

$$\simeq -2,49 \times 10^{-29} g^{-1} s$$

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$$\simeq -2,49 \times 10^{-29} g^{-1} s$$

- Noncommutative quantum mechanics

$$[Q^j, Q^k] = -2i\hbar\theta^{lk}$$

$$[Q^j, P_k] = i\hbar\delta^j{}_k$$

$$[P_j, P_k] = 0$$

Noncommutativity and gravity III

- Uncertainty relation

$$\Delta Q^1 \Delta Q^2 \geq \frac{1}{2}\hbar|\theta| \equiv: l_\theta^2$$

Noncommutativity and gravity III

- Uncertainty relation

$$\Delta Q^1 \Delta Q^2 \geq \frac{1}{2}\hbar|\theta| \equiv: l_\theta^2$$

- Numerical value for the earth planet

$$l_\theta \approx 10^{-28} cm$$

Noncommutativity and gravity IV

Table 1: The length l_θ for the planets of the solar system.

Planets	l_θ (cm)
Mercury	$3,80 \times 10^{-28}$
Venus	$1,17 \times 10^{-28}$
Earth	$1,15 \times 10^{-28}$
Mars	$3,87 \times 10^{-28}$
Jupiter	$9,72 \times 10^{-30}$
Saturn	$2,07 \times 10^{-29}$
Uranus	$6,29 \times 10^{-29}$
Neptune	$6,48 \times 10^{-29}$