# The uses of superprojectors

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# Irreducible representations of Poincaré Group

Eversince the work of Wigner it is clear that unitary irreducible representations of Poincaré group are labeled by a continous parameter $m$ a semi-integeer number $s$

# Fields and Induced representations

Let $R_A^B$ be an irreducible representation of the Lorentz group. Then we obtain (reductible) representations of Poincaré group:

$$R_A^B(\Lambda)F_B(\Lambda x + a) = F_A(x)$$

The set of fields that transform acording to a representation of the Lorentz group are too big. We need to set some restrictions in order to obtain an irreducible representation. If we are lucky we can use Casimirs of the group to select irreducible representations.

# Casimirs

In $D$ dimensions there are several casimirs that can be constructed form the dual of the Pauli-Lubasky four vector.

$$W_{\mu\nu\rho} = J_{[\mu\nu}P_{\rho]} = \frac{1}{3}\left(J_{\mu\nu}P_{\rho} + J_{\nu\rho}P_{\mu} + J_{\rho\mu}P_{\nu}\right)$$
$$W_{\mu\nu} = W_{\mu\nu\rho}P^{\rho}$$

This tensor commutes with $P_\mu$ so we can build Casimir invariants

$$W_2 = W_{\mu\nu}W^{\nu\mu} \qquad W_4 = W_{\mu_1\mu_2}W^{\mu_2\mu_3}W_{\mu_3\mu_4}W^{\mu_4\mu_1}$$
$$W_6 = W_{\mu_1\mu_2}\cdots W^{\mu_{12}\mu_1} \qquad W_8 = W_{\mu_1\mu_2}\cdots W^{\mu_{16}\mu_1}$$

# Projectors

Suppose we have a Casimir operator $K$ with a single different eigenvalue:

$$\mathbb{P} = \prod_{i \neq k} \frac{K - \lambda_i I}{\lambda_k - \lambda_i}$$

To be able to use this idea we need

- The representation we want is contained only one time
- The igenvalue $\lambda_k$ is different from the rest

# Examples

If $D = 4$ inside a field $B_\mu$ dwell a $s = 0$ representation and a $s = 1$ representation. The Projectors are

$$\mathbb{P}_{s=0} = \frac{-\partial_\mu \partial^\nu}{P^2}, \qquad \mathbb{P}_{s=1} = \frac{\partial_\mu \partial^\nu + P^2}{P^2}$$

We can use them to build covariant field equations with gauge invariance (yes, masive gauge invariance)

$$(P^2 + m^2)\mathbb{P}_{s=1} B_\mu = B_\mu$$
$$B_\mu \to B_\mu + \mathbb{P}_{s=0} \Lambda_\mu$$

Gauge invariance may be used to set $\partial^\mu B_\mu = 0$ and the remaining equation is, of course Proca equation.

The casimir in this case is Pauli-Lubansky four vector $W_2 = W^2$, whose eigenvalues are $m^2 j(j+1)$. Remember that

$$J_{\mu\nu} = L_\mu + S_{\mu\nu}, \qquad L_{\mu\nu} = X_\mu P_\mu - N_\nu P_\mu$$
$$(S^{\mu\nu})_{\alpha\beta} = i\left(\delta^\mu{}_\alpha \delta^\nu{}_\beta - \delta^\mu{}_\beta \delta^\nu{}_\alpha\right)$$

We may use this projectors to build non-local actions for every spin.

$$S = \int d^D x \Psi^A(x)(P^2 + m^2)\mathbb{P}\Psi$$

This is formal but we can make sense of some of this action introducing auxiliary fields.
I have not been able to make contact with Sinh-Hagen descriptions.
Non-local action for higher spin particles have been used by Francia and Sagnotti

# Clifford algebras, sucint introduction

Representations of Clifford algebras are $\Gamma_\mu$, $\mu = 0, .., D-1$

$$\{\Gamma_\mu, \Gamma_\nu\} = -2\eta_{\mu\nu}$$

For $D$ even we can find unitary matrices anti-diagonal matrices

$$\Gamma^\mu = \begin{pmatrix} 0 & \Sigma^\mu \\ \bar{\Sigma}^\mu & 0 \end{pmatrix}$$

where $\Sigma^0 = \bar{\Sigma}^0 = I$ and $\bar{\Sigma}^i = -\Sigma^i$. In odd dimensions we add

$$W = i^{d+1}\Gamma_0\Gamma_1 \cdots \Gamma_{D-1} \quad \Gamma^D = \pm iW$$

# Periodic properties

We construct these matrices by induction. Let $\tilde{\Gamma}^\mu$ the gamma matrices in dimension $D-2$, then

$$\Sigma^i = \tilde{\Gamma}^i \tilde{W} \qquad i = 1, 2, ..., D-3$$
$$\Sigma^{D-2} = i\tilde{W}\tilde{\Gamma}^0, \qquad \Sigma^{D-1} = \tilde{W}$$

matrices $B$, $C$, $\tilde{B}$, $\tilde{C}$ that make the transition:

$$B\Gamma^\mu B^{-1} = -(\Gamma^\mu)^* \qquad \tilde{B}\Gamma^\mu \tilde{B}^{-1} = (\Gamma^\mu)^*$$
$$C\Gamma^\mu C^{-1} = (\Gamma^\mu)^T \qquad \tilde{C}\Gamma^\mu \tilde{C}^{-1} = -(\Gamma^\mu)^T$$

There is only one independent matrix

$$B = \tilde{B}W, \; C = \tilde{C}W, \tilde{C} = \tilde{B}\Gamma^0$$

| $D$ | $BB^*$ | $\tilde{B}\tilde{B}^*$ | M | PM | SimM | PSimM | Sim | AntiSim |
|---|---|---|---|---|---|---|---|---|
| 2, 10 | $I$ | $I$ | Yes | Yes | No | No | $B, \tilde{B}, \tilde{C}$ | $C$ |
| 3, 11 | $I$ | – | Yes | – | No | – | $B$ | $C$ |
| 4 | $I$ | $-I$ | Yes | No | No | Yes | $B$ | $\tilde{B},\ C,\ \tilde{C}$ |
| 5 | – | $-I$ | – | No | – | Yes | | $\tilde{B},\ \tilde{C}$ |
| 6 | $-I$ | $-I$ | No | No | Yes | Yes | $C$ | $\tilde{B},\ B,\ \tilde{C}$ |
| 7 | $-I$ | – | No | – | Yes | – | | $\tilde{B},\ \tilde{C}$ |
| 8 | $-I$ | $I$ | No | Yes | Yes | No | $\tilde{B},\ C,\ \tilde{C}$ | $B$ |
| 9 | – | $I$ | – | Yes | – | No | $\tilde{B},\ \tilde{C}$ | |

# Supermultiplets

Irreducible representations of supersymmetry algebras are callled supermultiplets. We want to know what particles there are inside a given supermultiplet

| Susy/Spin | [0] | [1] | [2] | [3] | [4] |
|---|---|---|---|---|---|
| $N = 1$ | 2 | 1 | 0 | 0 | 0 |
| $N = 2$ | 4 | 5 | 1 | 0 | 0 |
| $N = 3$ | 14 | 14 | 6 | 1 | 0 |
| $N = 4$ | 42 | 48 | 27 | 8 | 1 |

Table: Masive supermultiplets in $D = 4$

# Massive supermultiplets in $D = 10$

| Irrep | Bosonic fields | Fermionic Fields |
|---|---|---|
| 1 | 44+84 | 128 |
| 9 | 9+36+126+156+231+594 | 16+128+432+576 |
| 16 | $9 + 36 + 44 + 84 + 126 + 231 + 594 + 924$ | $16 + 2 \times 128 + 432 + 576 + 768$ |
| 36 | $9 + 36 + 44 + 84 + 126 + 231 + 594 + 910 + 924 + 1650$ | $16 + 2 \times 128 + 432 + 576 + 768 + 2560$ |
| 44 | $1 + 36 + 44 + 84 + 231 + 45 + 495 + 924 + 2457$ | $16 + 128 + 432 + 576 + 1920 + 2560$ |
| 84 | $1 + 36 + 44 + 2 \times 84 + 126 + 231 + 495 + 594 + 2 \times 924 + 1980 + 2457 + 2772$ | $16 + 2 \times 128 + 2 \times 432 + 576 + 672 + 768 + 2560 + 5$ |
| 128 | $1 + 9 + 2 \times 36 + 44 + 2 \times 84 + 2 \times 126 + 156 + 2 \times 231 + 495 + 2 \times 594 + 910 + 2 \times 924 + 1650 + 2457 + 2772 + 3900$ | $2 \times 16 + 3 \times 128 + 3 \times 432 + 2 \times 576 + 672 + 768 + 2 \times 2560 + 5040$ |

# Superprojectors

## General theorem

The super spin content inside a superfield is the same as the spin content inside a supermultiplet.

# Superparticles

Here we use superprojectors to covariantly quantize superparticles (actually systems with second class constraints)
We start from an action for a massive superparticle in ten dimensions

$$S = \frac{1}{2}\int (e^{-1}\omega^\mu\omega^\nu\eta_{\mu\nu} - m^2 e)d\tau$$

with $\omega^\mu = \dot{x}^\mu + i\theta^a S^\mu{}_{ab}\dot{\theta}^b$. In this system we have a first class constraint and a family of second class constraints.

$$d_a = \pi_a + ip_\mu S^\mu{}_{ab}\theta^b$$
$$\{d_a, d_b\} = -2ip_\mu S^\mu{}_{ab}$$

# Second-class constraints

Poisson brackets with Dirac brackets.

## Non-commutative classical algebra

$$\left\{\theta^a, \theta^b\right\}_{\mathrm{D}} = \frac{i}{2p^2} p_\mu S^{\mu\, ab}$$

$$\{\theta^a, x^\mu\}_{\mathrm{D}} = \frac{1}{2p^2} \theta^b S^\mu{}_{bc} S^{\nu\, ca} p_\nu$$

$$\{x^\mu, x^\nu\}_{\mathrm{D}} = \frac{-\Sigma^{\mu\nu}}{p^2}$$

$$J^{\mu\nu} = L^{\mu\nu} + \Sigma^{\mu\nu}$$

$$L^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu \qquad \Sigma^{\mu\nu} = \frac{-1}{4} \theta S^{\mu\nu} \pi$$

$$\pi_a = -i p_\mu S^\mu{}_{ab} \theta^b$$

# Quantization

Straightforward canonical quantization now demands to switch Dirac brackets by commutators. So that our problem is now to find a set of operators that fulfil the quantum algebra

## Non-commutative quantum algebra

$$\left\{\hat{\Theta}^a, \hat{\Theta}^b\right\} = \frac{-1}{2P^2} P_\mu S^{\mu\, ab}$$

$$\left[\hat{X}^\mu, \hat{\Theta}^a\right] = \frac{i\hat{\Theta}^b}{2P^2} P_\nu S^\mu{}_{bc} S^{\nu\, ca}$$

$$\left[\hat{X}^\mu, \hat{X}^\nu\right] = \frac{-i\Sigma^{\mu\nu}}{P^2}$$

where $\Sigma^{\mu\nu}$ is the *internal* angular momentum given in this case by

$$\Sigma_{\mu\nu} = \frac{-1}{4}\hat{\Theta} S_{\mu\nu} \hat{\Pi}$$

# Using a superprojector

Now superprojectors come handy

**Theorem**

This algebra can be implemented at the quantum level if we find a superprojection operator $\mathbb{P}$ that meet the requirements

$$[\mathbb{P}, Q_a] = [\mathbb{P}, P_\mu] = [\mathbb{P}, J_{\mu\nu}] = 0$$
$$\mathbb{P} D_a \mathbb{P} = 0$$

Then a set of operators $(\hat{X}^\mu, \hat{\Theta}^a)$ that satisfy the quantum algebra of superspace would be given by the rule

$$\hat{X}^\mu = \mathbb{P} X^\mu \mathbb{P} \qquad \hat{\Theta}^a = \mathbb{P} \Theta^a \mathbb{P}$$

# Choosing a superprojector

We now have three such projectors and hence we have three representations of the algebra. The problem is that for these representations the internal angular momentum has a complex expression

$$\Sigma_{\mu\nu} = \frac{-1}{4}\hat{\Theta} S_{\mu\nu}\hat{\Pi} + T_{\mu\nu}$$

$$T_{\mu\nu} = \frac{-1}{4}\mathbb{P}\Theta S_{\mu\nu}\Pi\mathbb{P} + \frac{1}{4}\hat{\Theta} S_{\mu\nu}\hat{\Pi} = \frac{P_\alpha}{32P^2}\mathbb{P}DS^\alpha S^{\mu\nu}D\mathbb{P}$$

This extra term can also be written in terms of the operator defined earlier $C_{\mu\nu}$ as $T_{\mu\nu} = P^\alpha P_{[\alpha}C_{\mu\nu]}$. In four dimensions there are projectors (for a scalar superfield) such that $T_{\mu\nu} = 0$, but this is no longer true in $D = 10$. For the three projectors we have found the term $C_{\mu\nu}$ is strictly non-zero. If we want to realize the algebra we need to consider a *different* superfield. Now table comes handy. To get the correct algebra we need the smallest supermultiplet. The simplest superfield which contains such a representation is a symmetric, traceless and divergenceless tensor

# Non-local Lagrangian formulation