Bose-Einstein theory based in NCQM

J. Gamboa

Departamento de Física
Universidad de Santiago de Chile

São Paulo
September 8-12, 2009

- Magnetic-Dipole Spin Effects in Noncommutative Quantum Mechanics, (H. Falomir, J. Lopez, F. Mendez, P. Pisani and J.G), Phys. Lett. **B** in press.
- Towards to a Bose-Einstein theory for any integer spin (H. Falomir, F. Mendez and J.G), in progress.

plan

- Motivation
- Experimental sketch
- Basics NCQM
  Questions
- NCMQ and dipolar magnetics interactions
- First main conclusions
- Bose-Einstein and NC Hartree approach
- Spin and vortices
- Conclusions

Motivation

- BE condensation is a very old phenomenon observed in 1995.
- From the theoretical point of view the main assumption is

$$S = 0$$

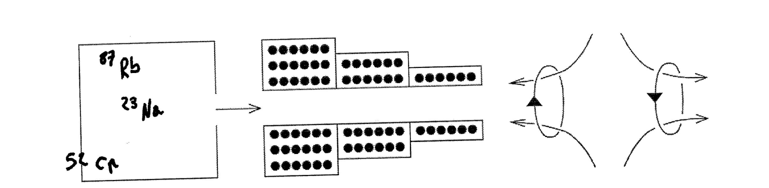
  Why?. large distances can be neglected.
- If we have two atoms with total spin $\mathbf{s}_1$ and $\mathbf{s}_2$ then the interaction potential at very large disntance is

$$V = \alpha \left(\frac{\mathbf{s}_1.\mathbf{s}_2 - 3(\mathbf{s}_1.\hat{\mathbf{r}})(\mathbf{s}_2.\hat{\mathbf{r}})}{r^3} \right), \qquad (1)$$

- However the dipolar interactions cannot be neglected even in the dilute approximation.
- Presently the experiments are complicated and the spin effects are no taken into account ....except recently (2006-..) (Stuttgart group).

Experimental sketch

A sketch of the experiment is



However one must

- ^{87}Rb and ^{23}Na were the original gas in the oven (1995) (S=1)
- ^{52}Cr, S=3
- In the original experiments spin was used in order to decelerate the atoms (Zeeman slower + an opposite laser beam)!!
- Atoms are captured in the magnetic-optical trap.

Questions

- If spin effects are explicitly taken into account there are differences but ...
- What happens with the “conventional vortex solution"?.
- The spin introduce a “spinor" order parameter in the Gross-Pitaevskii equation, are there new physical the effects?.

An alternative route; NCQM

NCQM means several things, *e.g.*, if we have $\hat{H}$ + commutators,

- First possibility

$$\begin{aligned} [x_i, p_j] &= i\delta_{ij}, \\ [x_i, x_j] &= i\theta_{ij}, [p_i, p_j] = iB_{ij} \end{aligned}$$

  where $\theta_{ij} = -\theta_{ji}$, $B_{ij} = -B_{ji}$ are constants matrices (Nair and Polychronakos PLB'2001).

- Second possibility

$$\begin{aligned} [x_i, p_j] &= i\delta_{ij}, \\ [x_i, x_j] &= i\theta^2 \epsilon_{ijk} s_k, \\ [p_i, p_j] &= 0 \end{aligned} \tag{2}$$

  and s_k is the spin operator and θ a parameter with dimensions of lenght.

- NC can be realized by using Bopp's shift (commutative variables)

$$x_i \rightarrow x_i + \frac{\theta_{ij}}{2} p_j$$

and NCQM becomes

$$H(x + \frac{\theta}{2} p, p)\psi = i\frac{\partial \psi}{\partial t},$$

(non-locality, ...)

- In the spin case one has the algebra (Snyder NR)

$$\begin{aligned} [\hat{x}_i, \hat{x}_j] &= i\theta^2 \epsilon_{ijk} \hat{s}_k, \\ [\hat{x}_i, \hat{p}_j] &= i\delta_{ij}, \qquad [\hat{p}_i, \hat{p}_j] = 0, \\ [\hat{x}_i, \hat{s}_j] &= i\theta \epsilon_{ijk} \hat{s}_k, \qquad [\hat{s}_i, \hat{s}_j] = i\epsilon_{ijk} \hat{s}_k, \end{aligned} \tag{3}$$

The analog of the Bopp's shift now is

$$\begin{aligned} \hat{x}_i &\quad \rightarrow \hat{x}_i = x_i + \theta s_i, \\ \hat{p}_i &\quad \rightarrow \hat{p}_i = p_i := -\imath \partial_i, \\ \hat{s}_i &\quad \rightarrow \hat{s}_i = s_i := \frac{\sigma_i}{2}, \end{aligned} \tag{4}$$

where x_i and p_i are now canonical operators satisfying the Heisenberg's algebra.

This simple observation implies that any noncommutative quantum mechanical system described by

$$\imath\partial_t \left|\psi(t)\right\rangle = \hat{H}(\hat{p},\hat{x},\hat{s}) \left|\psi(t)\right\rangle = \left[\frac{1}{2}\hat{p}^2 + \hat{V}(\hat{x})\right] \left|\psi(t)\right\rangle \qquad (5)$$

can equivalently be described by the *commutative* Schrödinger equation

$$\imath\partial_t \psi(\mathbf{x},t) = H(p_i, x_i + \theta s_i)\,\psi(\mathbf{x},t)\,, \qquad (6)$$

where $\psi(\mathbf{x},t)$ is a Pauli spinor.

An example: NC Harmonic oscillator

Hamiltonian

$$\hat{H} = -\frac{1}{2}\nabla^2 + \frac{1}{2}\hat{\mathbf{x}}^2, \\ = -\frac{1}{2}\nabla^2 + \frac{1}{2}\left(\mathbf{x} + \theta\mathbf{s}\right)^2 . \\ = -\frac{1}{2}\nabla^2 + \frac{1}{2}\mathbf{x}^2 + \theta\,\mathbf{s}.\mathbf{x} + \frac{\theta^2}{2}\,\mathbf{s}^2. \qquad (7)$$

dipolar interaction explicit!.

$\hat{H}$ is solved by using SSQM methods

- Ground state has no nodes

$$\tilde{H} = H - E_0 = Q_i^\dagger Q, \tag{8}$$

where

$$Q_i = \frac{1}{\sqrt{2}}\left(\partial_i + \hat{x}_i\right), \qquad Q_i^\dagger = \frac{1}{\sqrt{2}}\left(-\partial_i + \hat{x}_i\right), \tag{9}$$

- Following Zanelli and J.G (PLB 1985) one find

$$Q_i \to S \quad = \quad Q_i\psi_i = Q_i\sigma_i \otimes \sigma_- = Q \otimes \sigma_-, \tag{10}$$
$$Q_i^\dagger \to S^\dagger \quad = \quad Q_i^\dagger\psi_i^\dagger = Q_i^\dagger\sigma_i \otimes \sigma_+ = Q^\dagger \otimes \sigma_+, \tag{11}$$

The supersymmetric Hamiltonian is

$$H_s := \frac{1}{2}\{S^\dagger, S\} = \frac{1}{2}\, Q^\dagger Q \otimes \frac{\mathbf{1} + \sigma_{\mathbf{3}}}{2} + \frac{1}{2}\, QQ^\dagger \otimes \frac{\mathbf{1} - \sigma_{\mathbf{3}}}{2}\,, \qquad (12)$$

and this prescription obviously fulfills

$$\begin{aligned} [S, H_s] &= 0 = \left[S^\dagger, H_s\right], \\ \{S, S\} &= 0 = \{S^\dagger, S^\dagger\}\,. \end{aligned} \qquad (13)$$

Notice that this Hamiltonian is defined on a space of four component functions (a pair of spinors),

$$\psi = \begin{pmatrix} \Psi^{(1)} \\ \Psi^{(2)} \end{pmatrix}.$$

A straightforward calculation yields

$$H_s = \frac{1}{2}\left(-\frac{1}{2}\nabla^2 + \frac{1}{2}\mathbf{x}^2 + 3\ \theta\ \mathbf{x}\cdot\mathbf{s} + \frac{9}{8}\theta^2\right)\otimes 1_{2\times 2} - \frac{1}{2}\left(2\,\mathbf{s}\cdot\mathbf{L} + \frac{3}{2}\right)\otimes\sigma_3, \\ = H^{ss} + H_{NC}, \tag{14}$$

where

$$H^{ss} = \frac{1}{2}\left(-\frac{1}{2}\nabla^2 + \frac{1}{2}\mathbf{x}^2\right)\otimes 1_{2\times 2} - \frac{1}{2}\left(2\,\mathbf{s}\cdot\mathbf{L} + \frac{3}{2}\right)\otimes\sigma_3, \tag{15}$$

is the standard supersymmetric Hamiltonian in three-dimensions for the harmonic oscillator, whereas

$$H_{NC} = \frac{1}{2}\left(3\ \theta\ \mathbf{x}\cdot\mathbf{s} + \frac{9}{8}\theta^2\right)\otimes 1_{2\times 2}, \tag{16}$$

is the correction due to non-commutativity.
Actually, the term $\mathbf{x.s}$ is the dipole interaction mentioned above and $\frac{9}{8}\theta^2$ is just a correction to the ground state energy.

The ground state satisfy

$$S\psi_0 = 0 \qquad \text{or} \qquad S^\dagger \psi_0 = 0\,.$$

which implies that $\psi_0 = \begin{pmatrix} \Psi_0^{(1)} \\ \mathbf{0} \end{pmatrix}$, with

$$Q\,\Psi_0^{(1)} = Q_i \sigma_i \Psi_0^{(1)} = 0\,. \tag{17}$$

The general normalized solution is

$$\Psi_0^{(1)} = \pi^{-\frac{3}{4}}\, e^{\frac{3\theta}{2}\,\imath\,\mathbf{k}_I\cdot\mathbf{x}}\, e^{-\frac{1}{2}\left(\mathbf{x}-\frac{3\theta}{2}\mathbf{k}_R\right)^2} \chi_-(\hat{\mathbf{k}}), \tag{18}$$

with $\hat{\mathbf{k}} = \mathbf{k}_R + i\mathbf{k}_I$ is a complex unitary vector ($\hat{\mathbf{k}}^2 = 1 \;\Rightarrow\; {\mathbf{k}_R}^2 - {\mathbf{k}_I}^2 = 1\,,\; \mathbf{k}_R \cdot \mathbf{k}_I = 0$) and $\chi_-(\hat{\mathbf{k}})$ is a constant spinor satisfying

$$\left(\hat{\mathbf{k}} \cdot \boldsymbol{\sigma}\right) \chi_-(\hat{\mathbf{k}}) = -\chi_-(\hat{\mathbf{k}}),$$

with $\chi_-(\hat{\mathbf{k}})^\dagger \chi_-(\hat{\mathbf{k}}) = 1$

First main conclusions

- The ground state is infinitely degenerated
- The rotational symmetry is spontaneously broken
- $$< \mathbf{x} >_{\mathbf{k}} = \frac{3\theta}{2}\mathbf{k}_R$$
- $$< \mathbf{p} >_{\mathbf{k}} = \frac{3\theta}{2}\mathbf{k}_I$$

Bose-Einstein Condensation and NC Hartree Approach

One starting point:

$$H = \sum_{i=1}^{N}\left(-\frac{1}{2}\nabla^2 + u(\mathbf{r})\right) + \sum_{i>j} W(\mathbf{r}_i - \mathbf{r}_j).$$

where

$$W(\mathbf{r}_i - \mathbf{r}_j) = \gamma\,\delta(\mathbf{r}_i - \mathbf{r}_j)$$

and

$$\Psi \rightarrow \textit{Order}\ \text{Parameter}$$

- All the bosons are in the ground state (in BE phase $\Psi(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N)$)

$$\Psi(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N) = \prod_{i=1}^{N} \psi(\mathbf{r}_i),$$

- W becomes $(\hat{x} \rightarrow x + \theta s)$

$$\begin{aligned} W(\mathbf{r} - \mathbf{r}') &= \gamma\,\delta(\mathbf{r} - \mathbf{r}' + \theta(\mathbf{s} - \mathbf{s}')), \\ &= \gamma\,\delta(\mathbf{r} - \mathbf{r}') + \theta\gamma\Delta\mathbf{s}.\frac{\partial}{\partial \mathbf{r}}\delta(\mathbf{r} - \mathbf{r}') + \mathcal{O}(\theta^2). \end{aligned}$$

- Normalization condition

$$\int d^3 |\psi(\mathbf{r})|^2 = N.$$

- Tiny depletion

$$\int d^3 r \nabla n = 0,$$

with $n = |\phi|^2$.

Energy and Generalized Gross-Pitaevski Equations

Computing

$$\delta E = \delta \left(\frac{< \Psi | \neg \hat{H} | \Psi >}{< \Psi | \Psi >} \right) = 0,$$

$+$ Lagrange multipliers

$$E = N\,(E_{GP} + E_C) + \mathcal{O}(\theta^2),$$

where E_{GP} is the conventional Gross-Pitaevski term and

$$\begin{aligned} E_C &= \theta\,\omega^2 \int d^3r\,\mathbf{r}\,.\,\psi^*(\mathbf{r})\mathbf{s}\psi(\mathbf{r}) - \gamma\theta \int d^3r \nabla n\,.\,(\psi^*(\mathbf{r})\,\mathbf{s}\,\psi(\mathbf{r}))\,, \\ &\approx \theta\,\omega^2 \int d^3r\,\psi^*(\mathbf{r})\mathbf{r}\,.\,\mathbf{s}\psi(\mathbf{r}) + d.c + \mathcal{O}(\theta^2). \end{aligned}$$

Spin and Vortices

The Generalized Gross-Pitaevski Equations are straightforward for $s = 1$ and axial symmetry

$$\psi(\mathbf{r}) = \begin{pmatrix} \phi_1(\rho, z) e^{iq_1 \varphi} \\ \phi_0(\rho, z) e^{iq_0 \varphi} \\ \phi_{-1}(\rho, z) e^{iq_{-1} \varphi} \end{pmatrix} . \tag{19}$$

where q_0 and $q_\pm$ are winding numbers imply

$$\left[-\frac{1}{2} \left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2} \right) + \frac{q_0^2}{2\rho^2} + \frac{1}{2} \omega^2 \left(\rho^2 + z^2 \right) + g\phi^2 \right] \phi_0 = \mu \phi_0,$$

$$\left[-\frac{1}{2} \left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2} \right) + \frac{q_\pm^2}{2\rho^2} + \frac{1}{2} \omega^2 \left(\rho^2 + (z \pm \theta)^2 \right) + g\phi^2 \right] \phi_\pm = \mu \phi$$

and μ chemical potential.

Partial “Semi" conclusions

- NC can simulate physical effects (Landau)
- Nontrivial examples of QM (spontaneous rotational symmetry ...probably nontrivial fermionic systems)
- if $\phi^2 \approx \phi_0^2$ in first equation and son on $\rightarrow$ three independents vortices
- However stil we need numerical simulations and explain basic physics; vortex-vortex interaction $\neq 0$?, ...